

LAYER POTENTIALS C^* -ALGEBRAS OF DOMAINS WITH CONICAL POINTS

CATARINA CARVALHO AND YU QIAO

ABSTRACT. To a domain with conical points Ω , we associate a natural C^* -algebra that is motivated by the study of boundary value problems on Ω , especially using the method of layer potentials. In two dimensions, we allow Ω to be a domain with ramified cracks. We construct an explicit groupoid associated to $\partial\Omega$ and use the theory of pseudodifferential operators on groupoids and its representations to obtain our layer potentials C^* -algebra. We study its structure, compute the associated K -groups, and prove Fredholm conditions for the natural pseudodifferential operators affiliated to this C^* -algebra.

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INTRODUCTION

Let Ω be a bounded domain with conical points in \mathbb{R}^n , $n \geq 2$, that is, Ω is locally diffeomorphic to a cone with smooth, possibly disconnected, basis. To Ω , or more precisely to $\partial\Omega$, we associate a natural C^* -algebra, the *layer potentials C^* -algebra*, that is motivated by the study of boundary value problems on Ω , especially by applications of the method of layer potentials. In two dimensions, we allow Ω to be a domain with ramified cracks. The main aim of this paper is to prove Fredholm

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conditions for the natural pseudodifferential operators affiliated to the layer potentials C^* -algebra. We make use of pseudodifferential calculus on groupoids, so our first step is to associate to Ω a *boundary groupoid* and study its structure, as well as the structure of the resulting groupoid C^* -algebra. Moreover, we compute the associated C^* -algebraic K -groups and show that they depend only on the number of conical points. We expect our Fredholm criterion to have applications to boundary operators coming from the study of boundary value problems in Ω .

One of the classical approaches to solving boundary problems for (strongly) elliptic equations is via the method of layer potentials, which reduces differential equations to boundary integral equations. More explicitly, by reduction, we want to invert an operator of the form “ $\frac{1}{2} + K$ ” on appropriate boundary function spaces. For instance, if the boundary is \mathcal{C}^2 , then the relevant operator K is compact [16, 19] on $L^2(\partial\Omega)$. So the operator $\frac{1}{2} + K$ is Fredholm. Hence we can apply the classical Fredholm theory to the operator $\frac{1}{2} + K$ to solve the Dirichlet problem. If the boundary is \mathcal{C}^1 , then the integral operator K on $\partial\Omega$ is still compact [15] on $L^2(\partial\Omega)$, but that no longer holds if there are singularities on the boundary [13, 14, 18, 19, 24, 25, 33, 34, 35]. In the singular case, it is therefore natural to look for larger C^* -algebras containing these boundary integral operators and where we still have boundedness and Fredholm criteria. See, for instance, the survey [28], where the importance of understanding algebras of pseudodifferential operators on singular spaces is emphasized. In this paper, we tackle the case of conical domains, possibly with cracks and make some progress in this direction. Our approach is to construct a Lie groupoid associated to $\partial\Omega$ and take its convolution C^* -algebra, so that tools from analysis on groupoids become available.

We first consider a desingularization $\Sigma(\Omega)$ of Ω as in [5, 18, 30], which in this case basically amounts to replacing a, possibly disconnected, neighborhood of the conical points by a cylinder. One obtains in that way a manifold with corners, which moreover has the structure of a Lie manifold with boundary [4, 5]. In particular there are naturally defined Sobolev spaces behaving much as in the smooth boundary case, and nice regularity and Fredholmness criteria [4, 20, 21]. On the boundary, we have a decomposition

$$\partial\Sigma(\Omega) = \partial'\Sigma(\Omega) \cup \partial''\Sigma(\Omega),$$

where $\partial'\Sigma(\Omega)$ are the hyperfaces that correspond to faces of Ω and $\partial''\Sigma(\Omega)$ are the hyperfaces at infinity. The space $\partial'\Sigma(\Omega)$ can be identified with a desingularization of $\partial\Omega$. The layer potentials C^* -algebra is then defined as the C^* -algebra of a suitable Lie groupoid \mathcal{G} with units $\partial'\Sigma(\Omega)$ (Definition 4.1). We use the theory of pseudodifferential operators on groupoids to identify $C^*(\mathcal{G})$ with an algebra of operators. In fact, $C^*(\mathcal{G})$ is an ideal of the norm closure of the algebra of order zero pseudodifferential operators on \mathcal{G} . There is a bounded, injective representation π of the algebra of pseudodifferential operators on \mathcal{G} on $C_c^\infty(\partial'\Sigma(\Omega))$, mapping $C^*(\mathcal{G})$ to the compact operators. Moreover, the Sobolev spaces $H^m(\partial'\Sigma(\Omega))$, defined using the Lie structure, can be identified [5] with weighted Sobolev spaces $\mathcal{K}_a^m(\partial\Omega)$ on $\partial\Omega$, $a \in \mathbb{R}$ (see (10) for the precise definition). We show that Fredholm criteria for operators on groupoids, as in [20, 21], apply in our case, so that we obtain (Theorem 6.3) that Fredholmness of an operator

$$\pi(P) : \mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega) \rightarrow \mathcal{K}_{\frac{n-1}{2}}^0(\partial\Omega)$$

is equivalent to ellipticity and invertibility of a family of operators

$$P_x : H^m(\mathbb{R}^+ \times \partial\omega_p) \rightarrow L^2(\mathbb{R}^+ \times \partial\omega_p)$$

where p is a conical point, ω_p is the basis of the (local) cone at p , and x is at the boundary of $\partial'\Sigma(\Omega)$. We also obtain (Theorem 6.5) that we can replace the second condition above with invertibility of a family of Mellin convolution operators on $\mathbb{R}^+ \times \partial\omega_p$. In two dimensions, we allow our domain to have cracks (Corollary 6.4). Fredholm conditions of this form, which are often referred generally as full ellipticity, appear in many contexts related to index theory on singular spaces, see for instance [8, 12, 30, 31, 45, 50, 51, 52, 53] (and references therein).

When there are no singularities, the groupoid \mathcal{G} reduces to the pair groupoid of $\partial\Omega$ and the C^* -algebra $C^*(\mathcal{G})$ is then isomorphic to the algebra of compact operators on $\partial\Omega$. Consequently, the representation theory of $C^*(\mathcal{G})$ – in the case when there are no singularities – can be used to recover results from Fredholm theory.

In the case of a straight cone with basis $\omega \subset S^{n-1}$, the desingularization is $\Sigma(\Omega) = [0, \infty) \times \overline{\omega}$, and $\partial'\Sigma(\Omega) = [0, \infty) \times \partial\omega$. Our construction gives

$$\mathcal{G} = ([0, \infty) \rtimes \mathbb{R}^+) \times (\partial\omega)^2,$$

where $[0, \infty) \rtimes \mathbb{R}^+$ denotes the action groupoid and $(\partial\omega)^2$ the pair groupoid (see Section 3 for details). When the basis $\partial\omega$ of the cone at a singular point P is disconnected, which is always the case in two dimensions, for instance, the desingularization will associate several boundary faces to P . We allow here interaction between these faces at the groupoid level, in that there will be arrows between different connected components. This will be important in applications.

Since $\partial'\Sigma(\Omega)$ is a manifold with smooth boundary, one can also consider Melrose's b -calculus [30] to obtain a well-behaved class of pseudodifferential operators on $\partial'\Sigma(\Omega)$. Our pseudodifferential calculus contains the b -pseudodifferential operators, in that the boundary groupoid defined here contains the b -groupoid as an open subgroupoid. The main difference at the groupoid level is that in the usual b -calculus, there is no interaction (no arrows) between the different faces at P .

Groupoids and groupoid C^* -algebras have appeared useful in the analysis over singular spaces and, in particular, spaces with conical singularities, see for instance [1, 2, 9, 10, 11, 20, 21, 37, 39, 42].

In [9, 10], Debord and Lescure associated to a pseudomanifold M with a conical singularity a Lie groupoid which can be used to prove index theorems [10, 11] and to deal with elliptic theory on manifolds with conical singularities [23]. The general idea is to blow up the conical point to get a cylinder, then glue the pair groupoid of this cylinder and the tangent space of the smooth part of M in a way such that there is a natural smooth structure on the resulting groupoid. (This groupoid is called ‘a tangent space of M ’ in [10].)

On a different line, the C^* -algebra of the transformation groupoid (Example 1.7) $\mathcal{H} := [0, \infty) \rtimes \mathbb{R}^+$ is the algebra of Wiener-Hopf operators on \mathbb{R}^+ . In the work of Muhly and Renault [42], they used groupoid techniques to identify the C^* -algebra generated by Wiener-Hopf operators defined over polyhedral cones or homogeneous, self-dual cones, with the C^* -algebra of a locally compact measured groupoid. Their construction is motivated by the study of the structure of the C^* -algebra generated by multivariable Wiener-Hopf operators from the groupoid point of view. Moreover,

in [2] this groupoid approach was used to analyze the structure of the (generalized) Wiener-Hopf C^* -algebra and do index theory for Wiener-Hopf operators on cones.

The construction presented in this paper has different aims, in that it is motivated by PDE's and comes from the nature of the singularities. Our general purpose is that certain boundary convolution integral operators are in fact in the groupoid C^* -algebra, so we need to consider groupoids over the (desingularized) boundary of the conical domain. Furthermore, from this fact and the results presented here, we will be able to show that these integral operators are Fredholm between suitable weighted Sobolev spaces for domains with conical points of dimension greater than 3. However, for domains with cracks, the resulting layer potential operators are no longer Fredholm. All these issues will be analyzed thoroughly in a forthcoming paper.

Let us briefly review the contents of each section. In Section 1, we review some basic knowledge of Lie groupoids, define pseudodifferential operators on a Lie groupoid and, from this, we define the C^* -algebra of a Lie groupoid. In Section 2, we give the main concepts concerning domains with conical points, including the desingularization $\Sigma(\Omega)$ and the definition of weighted Sobolev spaces on Ω and $\partial\Omega$, which will be the natural domains of our operators. In Section 3, we consider the case of straight cones. We construct a canonical Lie groupoid over the desingularized boundary (a cylinder), whose C^* -algebra coincides with that of the Toeplitz / Wiener-Hopf operators. In Section 4, we generalize these constructions to the case of domains with conical points. We construct explicitly a canonical Lie groupoid \mathcal{G} associated to $\partial\Omega$, the so-called boundary groupoid, and study the properties of the groupoid C^* -algebra $C^*(\mathcal{G})$, which we dub the layer potentials C^* -algebra. In particular, we show that the boundary groupoid and the layer potentials C^* -algebra only depend, up to equivalence, on the number of singularities of the conical domain. In dimension two, we allow our domains to have ramified cracks; this case is considered in Section 4.2. In Section 5, we compute the K -theory of the layer potentials C^* -algebra $C^*(\mathcal{G})$ and of the indicial algebra at the boundary. Lastly, in Section 6, we obtain a Fredholm criterion for pseudodifferential operators on \mathcal{G} (including $C^*(\mathcal{G})$).

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1. PSEUDODIFFERENTIAL OPERATORS ON GROUPOIDS AND GROUPOID C^* -ALGEBRAS

1.1. Lie groupoids and Lie algebroids. In this subsection, we review some basic facts on Lie groupoids and Lie algebroids. We begin with the definition of groupoids.

Definition 1.1. A *groupoid* is a small category \mathcal{G} in which each arrow is invertible.

Let us make this definition more explicit [6, 21, 36, 48]. A groupoid \mathcal{G} consists of two sets, a set of objects (or units) \mathcal{G}_0 and a set of arrows \mathcal{G}_1 . Usually we shall denote the space of units of \mathcal{G} by M and we shall identify \mathcal{G} with \mathcal{G}_1 . Each object of \mathcal{G} can be identified with an arrow of \mathcal{G} . We have an injective map $u : M := \mathcal{G}_0 \rightarrow \mathcal{G}_1$, where $u(x)$ is the identity arrow of an object x . To each arrow $g \in \mathcal{G}$ we associate

two units: its domain $d(g)$ and its range $r(g)$. The multiplication $\mu(g, h) = gh$ of two arrows $g, h \in \mathcal{G}$ is not always defined; it is defined exactly when $d(g) = r(h)$. The multiplication is associative. The inverse of an arrow is denoted by $g^{-1} = \iota(g)$.

A groupoid \mathcal{G} is therefore completely determined by the sets $\mathcal{G}_0, \mathcal{G}_1$ and the structural maps d, r, μ, u, ι . Consequently, we sometimes denote $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, d, r, \mu, u, \iota)$. The structural maps satisfy the following properties:

- (1) $d(hg) = d(g), r(hg) = r(h)$,
- (2) $k(hg) = (kh)g$
- (3) $u(r(g))g = g = gu(d(g))$, and
- (4) $d(g^{-1}) = r(g), r(g^{-1}) = d(g), g^{-1}g = u(d(g))$, and $gg^{-1} = u(r(g))$

for any $k, h, g \in \mathcal{G}_1$ with $d(k) = r(h)$ and $d(h) = r(g)$. The structural maps in a groupoid \mathcal{G} together fit into a diagram [36]

$$\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \xrightarrow{\mu} \mathcal{G}_1 \xrightarrow{\iota} \mathcal{G}_1 \xrightleftharpoons[r]{d} \mathcal{G}_0 \xrightarrow{u} \mathcal{G}_1.$$

The following definition is taken from [21].

Definition 1.2. A *Lie groupoid* is a groupoid

$$\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, d, r, \mu, u, \iota)$$

such that $M := \mathcal{G}_0$ and \mathcal{G}_1 are smooth manifolds (with or without corners), the structural maps d, r, μ, u , and ι are smooth, the domain map d is a submersion, and all the spaces M and $\mathcal{G}_x = d^{-1}(x)$, $x \in M$, are Hausdorff.

We now recall the definition of a Lie algebroid [21].

Definition 1.3. A *Lie algebroid* A over a manifold M is a vector bundle A over M , together with a Lie algebra structure on the space $\Gamma(A)$ of the smooth sections of A and a bundle map $\rho : A \rightarrow TM$, extended to a map between sections of these bundles, such that

- (1) $\rho([X, Y]) = [\rho(X), \rho(Y)]$;
- (2) $[X, fY] = f[X, Y] + (\rho(X)f)Y$,

for all smooth sections X and Y of A and any smooth function f on M . The map ρ is called the *anchor*. Usually we shall denote by (A, ρ) such a Lie algebroid.

Consider a Lie groupoid \mathcal{G} with units M . We can associate a Lie algebroid $A(\mathcal{G})$ to \mathcal{G} as follows [27]. The d -vertical subbundle of $T\mathcal{G}$ for $d : \mathcal{G} \rightarrow M$ is denoted by $T^d(\mathcal{G})$ and called simply the d -vertical bundle for \mathcal{G} . It is an involutive distribution on \mathcal{G} whose leaves are the components of the d -fibers of \mathcal{G} . (Here involutive distribution means that $T^d(\mathcal{G})$ is closed under the Lie bracket, i.e. if $X, Y \in \mathfrak{X}(\mathcal{G})$ are sections of $T^d(\mathcal{G})$, then the vector field $[X, Y]$ is also a section $T^d(\mathcal{G})$.) Hence we obtain

$$T^d\mathcal{G} = \ker d_* = \bigcup_{x \in M} T\mathcal{G}_x \subset T\mathcal{G}.$$

The *Lie algebroid* of \mathcal{G} , denoted by $A(\mathcal{G})$, is defined to be $T^d(\mathcal{G})|_M$, the restriction of the d -vertical tangent bundle to the set of units M . In this case, we say that \mathcal{G} integrates $A(\mathcal{G})$.

Let $f_1, f_2 \in \mathcal{C}_c^\infty(\mathcal{G})$ and fix a Haar system of measures $d\mu_x$ on the d -fibers. The *convolution* product of f_1 and f_2 is defined as

$$f_1 * f_2(h) := \int_{d^{-1}(d(h))} f_1(g) f_2(g^{-1}h) d\mu_{d(h)},$$

and $\mathcal{C}_c^\infty(\mathcal{G})$ becomes a $*$ -algebra with $f^*(g) := \overline{f(g^{-1})}$. Taking the closure with respect to a suitable norm, given by the sup over all bounded representations, we get the *groupoid C^* -algebra* $C^*(\mathcal{G})$ (see [48]). For our purposes, it is preferable to define $C^*(\mathcal{G})$ as an algebra of operators on \mathcal{G} , as we shall see now (Definition 1.4 below).

1.2. Pseudodifferential operators and groupoid C^* -algebras. We recall here the construction of the space of pseudodifferential operators associated to a Lie groupoid \mathcal{G} with units M [20, 21, 38, 37, 40, 47]. The dimension of M is $n > 0$.

Namely, let $P = (P_x)$, $x \in M$ be a smooth family of pseudodifferential operators acting on \mathcal{G}_x . We say that P is *right invariant* if $P_{r(g)}U_g = U_gP_{d(g)}$, for all $g \in \mathcal{G}$, where

$$U_g : C^\infty(\mathcal{G}_{d(g)}) \rightarrow C^\infty(\mathcal{G}_{r(g)}), \quad (U_g f)(g') = f(g'g).$$

Let k_x be the distributional kernel of P_x , $x \in M$. Note that the support of the P

$$\text{supp}(P) := \overline{\bigcup_{x \in M} \text{supp}(k_x)} \subset \{(g, g'), d(g) = d(g')\} \subset \mathcal{G} \times \mathcal{G}$$

since $\text{supp}(k_x) \subset \mathcal{G}_x \times \mathcal{G}_x$. Let $\mu_1(g', g) := g'g^{-1}$. The family $P = (P_x)$ is called *uniformly supported* if its *reduced support* $\text{supp}_\mu(P) := \mu_1(\text{supp}(P))$ is a compact subset of \mathcal{G} .

Definition 1.4. The space $\Psi^m(\mathcal{G})$ of *pseudodifferential operators of order m on a Lie groupoid \mathcal{G}* with units M consists of smooth families of pseudodifferential operators $P = (P_x)$, $x \in M$, with $P_x \in \Psi^m(\mathcal{G}_x)$, which are uniformly supported and right invariant.

We also denote $\Psi^\infty(\mathcal{G}) := \bigcup_{m \in \mathbb{R}} \Psi^m(\mathcal{G})$ and $\Psi^{-\infty}(\mathcal{G}) := \bigcap_{m \in \mathbb{R}} \Psi^m(\mathcal{G})$. We then have a representation π of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(M)$ (or on $\mathcal{C}^\infty(M)$, on $L^2(M)$, or on Sobolev spaces), called *vector representation* uniquely determined by the equation

$$(1) \quad (\pi(P)f) \circ r := P(f \circ r),$$

where $f \in \mathcal{C}_c^\infty(M)$ and $P = (P_x) \in \Psi^m(\mathcal{G})$.

An alternative definition of $\Psi^m(\mathcal{G})$ is through distribution kernels (see for instance [4, 40, 47]). More precisely, $k_P(g) := k_{d(g)}(g, d(g))$ defines a distribution on \mathcal{G} , with $\text{supp} k_P = \text{supp}_\mu(P)$ compact, smooth outside M and given by an oscillatory integral on a neighborhood of M . We say that $k_P \in I_c^m(\mathcal{G}; M)$ is a conormal distribution to M . Conversely, we have $P_x f_x(g) = \int_{\mathcal{G}_x} k_P(gh^{-1}) f_x(h) d\mu_x(h)$, and $\Psi^m(\mathcal{G}) \cong I_c^m(\mathcal{G}; M)$. If $P \in \Psi^{-\infty}(\mathcal{G})$, then P identifies with the convolution with a smooth, compactly supported function and $\Psi^{-\infty}(\mathcal{G})$ identifies with the convolution algebra $\mathcal{C}_c^\infty(\mathcal{G})$. In particular, we can define

$$(2) \quad \|P\|_{L^1(\mathcal{G})} := \sup_{x \in M} \left\{ \int_{\mathcal{G}_x} |k_P(g^{-1})| d\mu_x(g), \int_{\mathcal{G}_x} |k_P(g)| d\mu_x(g) \right\}.$$

There is an interesting representation, the *regular representation* π_x , associated to x on $\mathcal{C}_c^\infty(\mathcal{G}_x)$, defined by $\pi_x(P) = P_x$. It is clear that $\|\pi_x(P)\| \leq \|P\|_{L^1}$. The *reduced C^* -norm* of P is defined by

$$(3) \quad \|P\|_r = \sup_{x \in M} \|\pi_x(P)\| = \sup_{x \in M} \|P_x\|,$$

and the *full norm* of P is defined by

$$(4) \quad \|P\| = \sup_{\rho} \|\rho(P)\|,$$

where ρ varies over all bounded representations of $\Psi^0(\mathcal{G})$ satisfying

$$\|\rho(P)\| \leq \|P\|_{L^1} \quad \text{for all } P \in \Psi^{-\infty}(\mathcal{G}).$$

Definition 1.5. Let \mathcal{G} be a Lie groupoid and $\Psi^\infty(\mathcal{G})$ be as above. We define $C^*(\mathcal{G})$ (respectively, $C_r^*(\mathcal{G})$) to be the closure of $\Psi^{-\infty}(\mathcal{G})$ in the norm $\|\cdot\|$ (respectively, $\|\cdot\|_r$). If $\|\cdot\|_r = \|\cdot\|$, that is, if $C^*(\mathcal{G}) \cong C_r^*(\mathcal{G})$, we call \mathcal{G} *amenable*.

Now consider the closure $\overline{\Psi^0(\mathcal{G})}$ with respect to $\|\cdot\|$. Let A denote the algebroid defined by \mathcal{G} , and S^*A denote the sphere bundle of A^* . There is a well-defined principal symbol mapping $\sigma : \overline{\Psi^0(\mathcal{G})} \rightarrow \mathcal{C}_0(S^*A)$, which is a surjective $*$ -homomorphism and its kernel coincides with $C^*(\mathcal{G})$:

$$(5) \quad 0 \longrightarrow C^*(\mathcal{G}) \longrightarrow \overline{\Psi^0(\mathcal{G})} \xrightarrow{\sigma_0} \mathcal{C}_0(S^*A) \longrightarrow 0,$$

In particular $\Psi^{-\infty}(\mathcal{G})$ is dense in $\Psi^{-1}(\mathcal{G})$ and $\Psi^{-1}(\mathcal{G}) \subset C^*(\mathcal{G})$. An operator $P \in \overline{\Psi^0(\mathcal{G})}$ is said to be *elliptic* if $\sigma_0(P)$ is invertible in $\mathcal{C}_0(S^*A)$.

Let $Y \subset M$ be an *invariant* subset, that is, such that $d^{-1}(Y) = r^{-1}(Y)$. Then, if Y is a closed submanifold of M , $\mathcal{G}_Y := d^{-1}(Y)$ is also a Lie groupoid, with units Y and there is an exact sequence

$$(6) \quad 0 \longrightarrow C^*(\mathcal{G}_{M \setminus Y}) \longrightarrow C^*(\mathcal{G}) \longrightarrow C^*(\mathcal{G}_Y) \longrightarrow 0.$$

Moreover, there is a restriction map $\mathcal{R}_Y : \Psi^m(\mathcal{G}) \rightarrow \Psi^m(\mathcal{G}_Y)$. In this case, Lemma 3 in [20] gives that the following sequence is exact:

$$(7) \quad 0 \longrightarrow C^*(\mathcal{G}_{M \setminus Y}) \longrightarrow \overline{\Psi^0(\mathcal{G})} \xrightarrow{(\mathcal{R}_Y, \sigma)} \overline{\Psi^0(\mathcal{G}_Y)} \times_{\mathcal{C}_0(S^*A_Y)} \mathcal{C}_0(S^*A) \longrightarrow 0,$$

where the fibered product $\overline{\Psi^0(\mathcal{G}_Y)} \times_{\mathcal{C}_0(S^*A_Y)} \mathcal{C}_0(S^*A)$ is defined as the algebra of pairs $(Q, f) \in \overline{\Psi^0(\mathcal{G}_Y)} \times \mathcal{C}_0(S^*A)$ such that $\sigma(Q) = f|_{S^*A_Y}$. One possible strategy to prove Fredholmness for pseudodifferential operators on \mathcal{G} , in particular, in $C^*(\mathcal{G})$, is to look for invariant subsets $Y \subset M$ such that the C^* -algebra of \mathcal{G}_Y is (isomorphic to) the compact operators. This is the case in the first example we consider below. See [20, 47] for details.

Example 1.6 (Pair groupoid). Let M be a smooth manifold (with or without corners). Let

$$\mathcal{G} = M \times M \quad \mathcal{G}_0 = M,$$

with structure maps $d(m_1, m_2) = m_2$, $r(m_1, m_2) = m_1$, $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$, $u(m) = (m, m)$, and $\iota(m_1, m_2) = (m_2, m_1)$. Then \mathcal{G} is a Lie groupoid, called the *pair groupoid*. We have $A(\mathcal{G}) = TM$. According to the definition, a pseudodifferential operator P belongs to $\Psi^m(\mathcal{G})$ if and only if the family $P = (P_x)_{x \in M}$ is constant. Hence we obtain $\Psi^m(\mathcal{G}) = \Psi_{\text{comp}}^m(M)$. Also, an important result is that $C^*(\mathcal{G}) \cong \mathcal{K}$, the ideal of compact operators, the isomorphism being given by

the vector representation or by any of the regular representations (together with $\mathcal{G}_x \cong M$.) If M has dimension 0, say, it is a discrete set with k elements, then $C^*(\mathcal{G}) \cong M_k(\mathbb{C})$ and the convolution product becomes matrix multiplication.

Example 1.7 (Transformation (or Action) groupoid). Suppose that a Lie group G acts on the smooth manifold M from the right. The *transformation groupoid* over $M \times \{e\} \cong M$, denoted by $M \rtimes G$, is the set $M \times G$ with structure maps $d(m, g) = (m \cdot g, e)$, $r(m, g) = (m, e)$, $(m, g)(m \cdot g, h) = (m, gh)$, $u(m, e) = (m, e)$, and $\iota(m, g) = (m \cdot g, g^{-1})$. For more on the action groupoid, one may see [27, 36, 48].

Let \mathfrak{g} be the Lie algebra of G . Denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M . The action of G on M induces a Lie algebra homomorphism $\phi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$, i.e., an action of the Lie algebra \mathfrak{g} on M . The *transformation Lie algebroid* $M \rtimes \mathfrak{g}$ has anchor map $\rho : M \rtimes \mathfrak{g} \rightarrow TM$ defined by

$$\rho(m, v) = \phi(v)(m).$$

Any section v of $M \rtimes \mathfrak{g}$ is a map $v : M \rightarrow \mathfrak{g}$. We define the bracket on sections of $M \rtimes \mathfrak{g}$ by

$$[v, w](m) = [v(m), w(m)]_{\mathfrak{g}} + (\phi(v(m)) \cdot w)(m) - (\phi(w(m)) \cdot v)(m).$$

For more details, see [6]. In general, there is no obvious description of pseudodifferential operators on transformation groupoid which depends on the action of G on M . One case of interest is $\mathcal{G} = [0, \infty) \rtimes \mathbb{R}^+$, where \mathbb{R}^+ acts by dilation. In this case, it is known that $C^*(\mathcal{G})$ coincides with the class of Wiener-Hopf operators and we have $C^*(\mathcal{G}) = \mathcal{C}_0([0, \infty)) \rtimes \mathbb{R}^+$ [42] (see also Example 5.11 and the proof of Lemma 10.2 in [21]). Moreover, the anchor map $\rho : [0, \infty) \times \mathbb{R}^+ \rightarrow T[0, \infty)$ is such that $\rho(0, \lambda) = 0$, for all $\lambda \in \mathbb{R}^+$ and is injective otherwise, so we have

$$\Gamma(A(\mathcal{H})) \cong \{a(x)x\partial_x, a \in C^\infty([0, \infty))\},$$

the vector fields that vanish at 0, that is, at the boundary.

The following example is of a different nature, in that our starting point is the Lie algebroid.

Example 1.8 (b -groupoid). Let M be a manifold with smooth boundary and let \mathcal{V}_b denote the class of vector fields on M that are tangent to the boundary. The following construction is due to Melrose and led to the general concept of b -geometry [30], and later to the more general definition of Lie manifolds. The associated groupoid was defined in [38, 47].

According to the Serre-Swan theorem ([17] see also [4, 31, 30]), there exists a smooth vector bundle ${}^bTM \rightarrow M$ together with a natural map of vector bundles

$$\begin{array}{ccc} {}^bTM & \xrightarrow{\rho} & TM \\ \downarrow & & \downarrow \\ M & \xlongequal{\quad} & M \end{array}$$

such that $\mathcal{V}_b = \rho(\Gamma({}^bTM))$. We call bTM the *b -tangent bundle*. Since the Lie bracket of vector fields tangent to a submanifold is again tangent to that submanifold, we see that \mathcal{V}_b is a Lie algebra and bTM becomes a Lie algebroid.

Let

$$\mathcal{G}_b := \bigcup_j \mathbb{R}^+ \times (\partial_j M)^2 \quad \bigcup \quad M_0^2,$$

where M_0^2 denotes the pair groupoid of $M_0 := \text{int}(M)$ and $\partial_j M$ denote the connected components of ∂M . Then \mathcal{G}_b can be given the structure of a Lie groupoid with units M and we have that it integrates ${}^b TM$, that is, $A(\mathcal{G}_b) = {}^b TM$. The pseudodifferential calculus obtained is Melrose's b -calculus. (In fact, Melrose's calculus is a little bit larger.) See [30, 38, 37, 47] for details.

2. DOMAINS WITH CONICAL POINTS, DESINGULARIZATION, AND SOBOLEV SPACES

We review here the main concepts needed regarding domains with conical singularities.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open connected bounded domain. We say that Ω is a *domain with conical points* if there exists a finite number of points $\{p_1, p_2, \dots, p_l\} \subset \partial\Omega$, such that

- (1) $\partial\Omega \setminus \{p_1, p_2, \dots, p_l\}$ is smooth;
- (2) for each point p_i , there exist a neighborhood V_{p_i} of p_i , a possibly disconnected domain $\omega_{p_i} \subset S^{n-1}$, $\omega_{p_i} \neq S^{n-1}$, with smooth boundary, and a diffeomorphism $\phi_{p_i} : V_{p_i} \rightarrow B^n$ such that

$$\phi_{p_i}(\Omega \cap V_{p_i}) = \{rx' : 0 < r < 1, x' \in \omega_{p_i}\}.$$

If, moreover, $\partial\Omega = \partial\overline{\Omega}$, then we say that Ω is a *domain with no cracks*. The points p_i , $i = 1, \dots, l$ are called *conical points* or *vertices*. If $n = 2$, Ω is said to be a *polygonal domain*.

(We can always assume that $\overline{V_i} \cap \overline{V_j} = \emptyset$, for $i \neq j$, $i, j \in \{1, 2, \dots, l\}$, and that $J\phi_k(0) = I_n$, where $J\phi(0)$ is the Jacobian matrix of ϕ_i at p_i .)

Remark 2.2. Let $V = \{p_1, p_2, \dots, p_l\}$ be the given set of conical points of a conical domain with no cracks Ω , as above. The set V does not determine the structure of Ω since we can always increase it, but the minimum set of conical points is unique and coincides with the singularities of $\partial\Omega$. These are *true conical points* of Ω . The other points in V will be called *artificial points* and are the ones for which ω_{p_i} is diffeomorphic to a hemisphere S_+^{n-1} . (See Remark 1 in [29].) Note that in fact for any $x \in \overline{\Omega}$, we can take a neighborhood V_x of x , a domain $\omega_x \subset S^{n-1}$ and a diffeomorphism ϕ_x such that $\phi_{p_i}(\Omega \cap V_{p_i}) = \{rx' : 0 < r < 1, x' \in \omega_{p_i}\}$. Then $x \in \Omega$ if, and only if, $\omega_x = S^{n-1}$, and x is a smooth boundary point if, and only if, $\omega_x \cong S_+^{n-1}$. Otherwise, x is a true conical point. It is often useful in applications to boundary value problems on Ω to regard smooth boundary points as (artificial) vertices, representing for instance a change in boundary conditions.

The condition $\partial\Omega = \partial\overline{\Omega}$ means that no boundary point of Ω becomes an interior point of the closure, that is, all boundary points of Ω are accessible from the outside [16]. We call $x \in \partial\Omega$ a *crack point* if, using the notation in the previous remark, $\partial\omega_x \neq \partial(\overline{\omega_x})$. In this setting, x is a smooth boundary point if, and only if, $\omega_x \cong S_+^{n-1}$ or $\omega_x \cong S_+^{n-1} \sqcup S_+^{n-1}$, in which case Ω lies on both sides of $\partial\Omega$ close to x . We call x a *smooth crack point*. The remaining crack points are singular points of the boundary of the conical domain, that is, are true vertices. In Section 4.2, we will allow polygonal domains to have cracks (Definition 4.6). Note that in higher dimensions, domains with cracks will have edges, and thus are no longer conical

domains (in dimension two, the edges of the crack curves behave like conical points, so our constructions apply). See also [26, 29].

For the remainder of this section, we shall denote by Ω a bounded domain with conical points in \mathbb{R}^n , with *no cracks*. We now define the desingularization $\Sigma(\Omega)$ of Ω and $\partial\Omega$, which will play a major role in our constructions. We follow the approach in [5] (Section 4 and Example 2.11), see also [18, 30]. Let $\Omega^{(0)} = \{p_1, p_2, \dots, p_l\}$ be the set of conical points of Ω and ϕ_i, ω_i be as in Definition 2.1, for $i = 1, \dots, l$. Let also, for each i , ψ_i denote a smooth function on Ω such that $0 \leq \psi_i \leq 1$, $\psi_i = 1$ on $\phi^{-1}(\{rx' : 0 < r < \varepsilon_i, x' \in \omega_{p_i}\})$, for some $\varepsilon_i < 1$, and $\psi_i = 0$ outside $V_i \cap \Omega$. Define a map $\Phi : \Omega \setminus \Omega^{(0)} \rightarrow \mathbb{R}^{2n}$ by

$$\Phi(x) := \left(x, \sum_i \psi_i(x) |x - p_i|^{-1} x \right).$$

Then $\Sigma(\Omega)$ is defined as the closure in \mathbb{R}^{2n} of the image of Φ . The desingularization map $\kappa : \Sigma(\Omega) \rightarrow \overline{\Omega}$ is given by projection on the first n components. Note that $\kappa^{-1}(p_i) = \{p_i\} \times \overline{\omega_i}$. Note also that if $x \in V_i \cap \Omega$ is identified with rx' , $r \in (0, \varepsilon_i)$, $x' \in \omega_i$, then $\Phi(x) = (rx', x')$ and we have that $\Phi(V'_i \cap \Omega) \cong (0, \varepsilon_i) \times \omega_i$, for some $V'_i \subset V_i$, open. We have then the following isomorphism:

$$(8) \quad \Sigma(\Omega) \cong \left(\coprod_{p_i \in \Omega^{(0)}} [0, \varepsilon_{p_i}) \times \overline{\omega_{p_i}} \right) \bigcup_{\phi_{p_i}, p_i \in \Omega^{(0)}} \Omega,$$

where the two sets are glued by ϕ_i along $V'_i \cap \Omega$. So we see that $\Sigma(\Omega)$ is obtained from Ω by removing a (possibly non-connected) neighborhood of the singular points and replacing (each connected component) it by a cylinder. The advantage of using the approach above is that the results in [5] become available, in particular, the fact that $\Sigma(\Omega)$ is a Lie manifold with boundary, that is, a compact Riemannian manifold with corners with a given Lie algebra of vector fields tangent to the boundary defining the metric, which enjoys many of the properties of the smooth case [3, 4].

Note that it follows from (8) that the boundary is given by

$$(9) \quad \partial\Sigma(\Omega) \cong \left(\coprod_{p_i \in \Omega^{(0)}} [0, \varepsilon_{p_i}) \times \partial\omega_{p_i} \cup \{0\} \times \overline{\omega_{p_i}} \right) \bigcup_{\varphi_{p_i}, p_i \in \Omega^{(0)}} \Omega_0.$$

where Ω_0 denotes the smooth part of $\partial\Omega$, that is, $\Omega_0 := \partial\Omega \setminus \Omega^{(0)}$. There are different types of hyperfaces of $\partial\Sigma(\Omega)$. Namely, each face of Ω yields a hyperface, and to each p_i we have hyperfaces $\{0\} \times \overline{\eta_{ij}}$ and $[0, \varepsilon_{p_i}) \times \chi_{ik}$, where η_{ij} and χ_{ik} denote connected components of ω_i and $\partial\omega_i$, respectively. In the terminology of [4], the hyperface $[0, \infty) \times \overline{\chi_{ik}}$ is *not at infinity* since it corresponds to an actual face of Ω . The hyperface $\{0\} \times \overline{\eta_{ij}}$ is an hyperface *at infinity* because it corresponds to a singularity of Ω . We have a decomposition

$$\partial\Sigma(\Omega) = \partial'\Sigma(\Omega) + \partial''\Sigma(\Omega)$$

where $\partial'\Sigma(\Omega)$ denotes the union of hyperfaces which are not at infinity and $\partial''\Sigma(\Omega) := \kappa^{-1}(\Omega^{(0)})$ denotes the union of the hyperfaces at infinity. We remark that in fact $\partial'\Sigma(\Omega)$ can be identified with the desingularization of $\partial\Omega$ and $\partial'\Sigma(\Omega)$ is a Lie manifold (without boundary) [5]. (Note that our definition of $\partial'\Sigma(\Omega)$ differs from the one in [5], in that here we are considering the closure.)

The space $L^2(\Sigma(\Omega))$ is defined using the volume element of a compatible metric with the Lie structure at infinity on $\Sigma(\Omega)$. A compatible metric is $r_\Omega^{-2}g_e$, where g_e is the Euclidean metric and r_Ω is a weight function as in [5], representing the distance to the singular points. Then the Sobolev spaces $H^m(\Sigma(\Omega))$ are defined using $L^2(\Sigma(\Omega))$. It happens that these Sobolev spaces are related to *weighted* Sobolev spaces on Ω and $\partial\Omega$.

Let $m \in \mathbb{Z}_{\geq 0}$, α be a multiindex, and r_Ω be a weight function as in [5]. The m -th Sobolev space on Ω with weight r_Ω and index a is defined by

$$(10) \quad \mathcal{K}_a^m(\Omega) = \{u \in L_{\text{loc}}^2(\Omega), r_\Omega^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \leq m\}.$$

The following result can be found in [5] (Proposition 5.7 and Definition 5.8)

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain with conical points and $\Sigma(\Omega)$ be its desingularization. Let also $\partial'\Sigma(\Omega)$ be the union of the hyperfaces that are not at infinity. Then,*

- (a) $\mathcal{K}_{\frac{n}{2}}^m(\Omega) \cong H^m(\Sigma(\Omega), g)$, for all $m \in \mathbb{Z}$.
- (b) $\mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega) \cong H^m(\partial'\Sigma(\Omega))$, for all $m \in \mathbb{Z}_{\geq 0}$.

Note that the weighted Sobolev spaces on the boundary $\partial\Omega$ are defined using the identification given in (a) in the above proposition. For more details, see [5].

From the previous proposition, it follows that results on regularity and Fredholmness of operators on $\partial'\Sigma(\Omega)$ translate into results for boundary operators.

3. LIE ALGEBROIDS AND LIE GROUPOIDS FOR STRAIGHT CONES

Let $\omega \subset S^{n-1}$ be an open subset with smooth boundary. We allow ω to be *disconnected*. Denote by $\Omega := \{ty', y' \in \omega, t \in (0, \infty)\} = \mathbb{R}^+\omega$ the cone with base ω . We use here the concepts reviewed in the previous sections to place the approach in [46] in a setting easier to generalize to domains with conical points.

We first desingularize our domain with conical points Ω and its boundary $\partial\Omega$ as in the previous section. We obtain a Lie manifold $\Sigma(\Omega)$, which in this case coincides with an half-infinite solid cylinder

$$\Sigma(\Omega) = [0, \infty) \times \overline{\omega}$$

with boundary $\partial\Sigma(\Omega) = [0, \infty) \times \partial\omega \cup \{0\} \times \omega$. We denote by $\partial'\Sigma(\Omega) = [0, \infty) \times \partial\omega$ the union of the hyperfaces not at infinity and by $\partial''\Sigma(\Omega) = \{0\} \times \omega$ the union of the hyperfaces at infinity (recall that $\omega, \partial\omega$ might be disconnected). Throughout this section, we let $M := \partial'\Sigma(\Omega) = [0, \infty) \times \partial\omega$, which we think of as a desingularization of $\partial\Omega$.

The Lie algebra of vector fields $\mathcal{V}(\Omega)$ inducing the Lie structure at infinity in $\Sigma(\Omega)$ is the space of vector fields on $\Sigma(\Omega)$ that are tangent to $\{0\} \times \overline{\omega}$, that is, to the boundary at infinity [5, 29]. There is an induced Lie structure at infinity on M , that can be described as follows. If we denote by $p_1 : M \rightarrow [0, \infty)$ and $p_2 : M \rightarrow \partial\omega$ the two projections and decompose $TM = p_1^*T[0, \infty) \oplus p_2^*T\partial\omega$, we have that \mathcal{W} consists of vector fields on $M = [0, \infty) \times \partial\omega$ such that

$$\mathcal{W} = \{a(x, y)x\partial_x + Y, a \in C^\infty(M), Y \in \Gamma(p_2^*T\partial\omega)\},$$

that is, that are tangent to $\{0\} \times \partial\omega = \partial M$. Hence, the Lie algebroid associated to M as a Lie manifold is (isomorphic to) the b -tangent bundle bTM such that $\Gamma({}^bTM) = \mathcal{W}$ (see Example 1.8).

Let us consider now the action of $(0, \infty)$ on $[0, \infty)$ by dilation and let

$$(11) \quad \mathcal{H} := [0, \infty) \rtimes (0, \infty)$$

be the transformation (or action) groupoid (Example 1.7). Also, we denote by $X^2 := X \times X$ the pair groupoid with units some arbitrary space X (Example 1.6) and by \mathcal{K} the C^* -algebra of compact operators on some generic separable Hilbert space. Then $C^*(X^2) \simeq \mathcal{K}$, if $\dim X > 0$, and $C^*(X^2) \simeq M_k(\mathbb{C})$, if X is discrete with k elements.

We define the *boundary groupoid associated to a straight cone* $\Omega = \mathbb{R}^+\omega$ as the product Lie groupoid with units $M = [0, \infty) \times \partial\omega$, corresponding to a desingularization of $\partial\Omega$, as

$$(12) \quad \mathcal{J} := \mathcal{H} \times (\partial\omega)^2.$$

Noting that $(0, \infty)$ is an invariant subset where \mathcal{H} coincides with the pair groupoid, we have that $M_0 := \text{int}(M) = (0, \infty) \times \partial\omega$ and $\partial M = \{0\} \times \partial\omega$ are invariant subsets of M with respect to \mathcal{J} and we have

$$\mathcal{J}_{M_0} = ((0, \infty) \times \partial\omega)^2 = M_0^2, \quad \mathcal{J}_{\partial M} = (0, \infty) \times (\partial\omega)^2.$$

(Note that M_0 is really the smooth part of $\partial\Omega$.)

We see now that \mathcal{J} integrates bTM , that is, that the sections of $A(\mathcal{J})$ are vector fields tangent to ∂M :

Lemma 3.1. *The Lie algebroid of \mathcal{J} is isomorphic to bTM , the b -tangent bundle.*

Proof. We use the general fact that the Lie algebroid of the product groupoid $\mathcal{G}_1 \times \mathcal{G}_2$ is the product of their Lie algebroids, i.e. $A(\mathcal{G}_1 \times \mathcal{G}_2) = A(\mathcal{G}_1) \times A(\mathcal{G}_2)$ (see Example 2.5 in [21]). We have that (see Example 1.7)

$$\Gamma(A(\mathcal{H})) \cong \mathcal{V} := \{a(x)x\partial_x, a \in C^\infty([0, \infty))\},$$

that is, the Lie algebroid of \mathcal{H} is isomorphic to the one coming from \mathcal{V} through the Serre-Swan theorem. Since $A((\partial\omega)^2) = T(\partial\omega)$, we get

$$\Gamma(A(\mathcal{J})) = \Gamma(A(\mathcal{H}) \times T(\partial\omega)) \cong \mathcal{W} = \Gamma({}^bTM)$$

and the conclusion follows from the Serre-Swan theorem. \square

Remark 3.2. Recall the definition of b -groupoid in Example 1.8, which, in the case of $M = [0, \infty) \times \partial\omega$ comes down to

$${}^b\mathcal{G} = M_0^2 \bigcup_j \mathbb{R}^+ \times (\partial_j\omega)^2,$$

where $\partial_j\omega$ denote the connected components of $\partial\omega$. If $\partial\omega$ is connected, then $\mathcal{J} = {}^b\mathcal{G}$. In the more interesting case of $\partial\omega$ being disconnected, the groupoid \mathcal{J} is larger and not d -connected. (Note that if $n = 2$, that is, if we have a polygonal domain, then $\partial\omega$ is *always* disconnected.) The main difference is that here we allow the different connected components of the boundary, corresponding to faces in the desingularization, to interact, in that there are arrows between them. In fact, we have that $\mathbb{R}^+ \times (\partial_j\omega)^2$ is a connected component of $\mathcal{J}_{\partial M} = \mathbb{R}^+ \times (\partial\omega)^2$, and we have $\mathbb{R}^+ \times (\partial_j\omega)^2 = \mathcal{J}_{\partial_j\omega}^{\partial_j\omega}$, the subgroupoid over $\partial_j\omega$ defined as $d^{-1}(\partial_j\omega) \cap r^{-1}\partial_j\omega$. Hence ${}^b\mathcal{G}$ is an open, wide subgroupoid of \mathcal{J} .

As for C^* -algebras, it is known that $C^*(\mathcal{H}) = \mathcal{C}_0([0, \infty)) \rtimes \mathbb{R}^+$ [42]. We have then the following:

Lemma 3.3. *Let $\mathcal{J} = \mathcal{H} \times (\partial\omega)^2$. Then $C^*(\mathcal{J}) \cong \mathcal{C}_0([0, \infty)) \rtimes \mathbb{R}^+ \otimes \mathcal{K}$, if $n \geq 3$, and $C^*(\mathcal{J}) \cong \mathcal{C}_0([0, \infty)) \rtimes \mathbb{R}^+ \otimes M_k(\mathbb{C}) = M_k(C^*(\mathcal{H}))$, if $n = 2$, where k is the number of elements of $\partial\omega$.*

Proof. The C^* -algebra of the pair groupoid $(\partial\omega)^2$ is isomorphic to \mathcal{K} , if $n \geq 3$, and to $M_k(\mathbb{C})$, if $n = 2$, where k is the number of elements of $\partial\omega$. Since (see Proposition 4.5 in [21])

$$C^*(\mathcal{J}) \cong C^*(\mathcal{H}) \otimes C^*(\partial\omega \times \partial\omega),$$

the result follows. \square

The C^* -algebra associated to \mathcal{H} is the algebra of Wiener-Hopf operators on \mathbb{R}^+ , and its unitalization is the algebra of Toeplitz operators [42]. In particular, we see that the groupoid C^* -algebra of a straight cone is stably isomorphic to the C^* -algebra of Wiener-Hopf operators. Let us recall the Toeplitz exact sequence

$$0 \longrightarrow \mathcal{K}(L^2(\mathbb{R}^+)) \longrightarrow C^*(\mathcal{H}) \longrightarrow \mathcal{C}_0((0, \infty)) \longrightarrow 0.$$

Tensoring with \mathcal{K} , we obtain another short exact sequence, for $n \geq 3$,

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(\mathcal{J}) \longrightarrow \mathcal{C}_0((0, \infty)) \otimes \mathcal{K} \longrightarrow 0$$

and similarly for $n = 2$. In fact, the sequence above can be obtained from (6), for the invariant, closed subset ∂M (see Proposition 4.2 below).

As for pseudodifferential operators, note that if $P \in \Psi^m(\mathcal{J})$, then $P_{M_0} \in \Psi^m(M_0^2)$, so that it identifies, by invariance, with an operator in $\Psi^m(M_0) = \Psi^m(\mathbb{R}^+ \times \partial\omega)$. At the boundary, $P_{\partial M} \in \Psi^m(\mathbb{R}^+ \times (\partial\omega)^2)$ is defined by a distribution kernel k_P in $\mathbb{R}^+ \times (\partial\omega)^2$. If $P_{\partial M} \in \Psi^{-\infty}(\mathbb{R}^+ \times (\partial\omega)^2)$, that is, if k_P is smooth, then it defines a smoothing *Mellin convolution operator* on $\mathbb{R}^+ \times \partial\omega$ (see [25, 46]). This is in fact one of the reasons behind our definition of \mathcal{J} .

Moreover, it follows from Remark 3.2 that $\Psi^m(\mathcal{J}) \supset \Psi({}^b\mathcal{G})$, Melrose's b -calculus. If we let $P = (P_x) \in \Psi^m(\mathcal{J})$, and, for $x \in \partial M$, write $P_x \in \Psi^m(\mathbb{R}^+ \times \partial\omega)$ as matrix of operators $P_x^{ij} : \mathcal{C}_c^\infty(\mathbb{R}^+ \times \partial_j\omega) \rightarrow \mathcal{C}_c^\infty(\mathbb{R}^+ \times \partial_i\omega)$, then the b -pseudodifferential operators at the boundary correspond to the diagonal entries. We call $\Psi^\infty(\mathcal{J}_{\partial M})$ the *indicial algebra*, in analogy with the usual indicial algebra for the b -calculus (which coincides with $\Psi^\infty({}^b\mathcal{G}_{\partial M})$) [30, 32, 38].

In the following section, we generalize these constructions to the setting of domains with conical points.

4. BOUNDARY GROUPOIDS FOR DOMAINS WITH CONICAL POINTS

4.1. Groupoid construction for domains with no cracks. We consider now a bounded domain Ω with conical points as in Definition 2.1. Our goal here is to define a groupoid over the desingularization of $\partial\Omega$, generalizing Equation (12). For now, Ω is not allowed to have cracks, but we will drop that assumption in the following subsection.

Let $\Omega^{(0)} = \{p_1, p_2, \dots, p_l\}$ be the set of conical points of Ω (for simplicity, we assume that all the p_i 's are true conical points). Denote by Ω_0 the smooth part of $\partial\Omega$, that is, $\Omega_0 = \partial\Omega \setminus \Omega^{(0)}$. Suppose that $p_i \in \Omega^{(0)}$ is a conical point. Then there exist a neighborhood $V_i \subset \mathbb{R}^n$ of p_i and a diffeomorphism $\phi_i : V_i \rightarrow B(0, 1) \subset \mathbb{R}^n$,

such that, up to a local change of coordinates, $J\phi_k(0) = I_n$, where $J\phi(0)$ is the Jacobian matrix of ϕ_i at p_i , and

$$\phi_i(V_i \cap \Omega) = B(0, 1) \cap \mathbb{R}_+\omega_i,$$

for some smooth domain $\omega_i \subset S^{n-1}$. Furthermore, we can assume that $\overline{V_i} \cap \overline{V_j} = \emptyset$, for $i \neq j$, $i, j \in \{1, 2, \dots, l\}$. As before, we denote by $\Sigma(\Omega)$ the manifold with corners obtained from the desingularization of Ω , as in Section 2 and by $M := \partial'\Sigma(\Omega)$ the union of the hyperfaces not at infinity, which we can identify with the desingularization of $\partial\Omega$.

We now construct a groupoid from Ω and $\partial\Omega$. From the discussion in Section 3, for each conical point $p_i \in \partial\Omega$, we can construct groupoid $\mathcal{J}_{p_i} = \mathcal{H} \times (\partial\omega_{p_i})^2$, where $\mathcal{H} = [0, \infty) \rtimes (0, \infty)$. Since the transformation groupoid $(0, \infty) \rtimes (0, \infty)$ is isomorphic to the pair groupoid $(0, \infty) \times (0, \infty)$ and $[0, \infty) \times (0, \infty)$ is diffeomorphic to $[0, \epsilon_{p_i}) \times (0, \epsilon_{p_i})$, for any $0 < \epsilon_{p_i} < 1$, we can impose a groupoid structure on $\mathcal{J}'_{p_i} := ([0, \epsilon_{p_i}) \times (0, \epsilon_{p_i})) \times (\partial\omega_{p_i} \times \partial\omega_{p_i})$ such that

- (a) the unit space M'_{p_i} of \mathcal{J}'_{p_i} is $[0, \epsilon_{p_i}) \times \partial\omega_{p_i}$,
- (b) $[0, \epsilon_{p_i}) \times (0, \epsilon_{p_i})$ has the same transformation groupoid structure as that of $[0, \infty) \rtimes (0, \infty)$;
- (c) the interior of \mathcal{J}'_{p_i} is isomorphic to the pair groupoid of $(0, \epsilon_{p_i}) \times \partial\omega_{p_i}$.

Let Ω_0 be the smooth part of $\partial\Omega$ as above and Ω_0^2 be the pair groupoid. For each $p_i \in \Omega^{(0)}$, there is a map $(0, \epsilon_{p_i}) \times \partial\omega_{p_i} \rightarrow \Omega_0$, $(t, x) \mapsto \phi_i^{-1}(xt)$, and it follows from (c) that we can define $\varphi_{p_i} : \text{int}(\mathcal{J}'_{p_i}) \rightarrow \Omega_0^2$ by

$$\varphi_{p_i}(t, s, \omega_1, \omega_2) = (\phi_i^{-1}(t\omega_1), \phi_i^{-1}(s\omega_2)),$$

where $t, s \in (0, \epsilon_i)$ and $\omega_1, \omega_2 \in \partial\omega_{p_i}$. It is clear that φ_{p_i} is smooth, a diffeomorphism into its image, and preserves the groupoid structure of \mathcal{J}'_{p_i} and Ω_0^2 . We can glue \mathcal{J}'_{p_i} and Ω_0^2 using the function φ_{p_i} , and we define the *boundary groupoid associated to a conical domain Ω* (with no cracks) as

$$(13) \quad \mathcal{G} := \left(\coprod_{p_i \in \Omega^{(0)}} \mathcal{J}'_{p_i} \right) \bigcup_{\varphi} \Omega_0^2,$$

where $\varphi = (\varphi_{p_i})_{p_i \in \Omega^{(0)}}$. Then \mathcal{G} is a Lie groupoid, with space of units

$$(14) \quad M = \left(\coprod_{p_i \in \Omega^{(0)}} [0, \epsilon_{p_i}) \times \partial\omega_{p_i} \right) \bigcup_{\varphi} \Omega_0 \cong \partial'\Sigma(\Omega),$$

where $\partial'\Sigma(\Omega)$ denotes the union of hyperfaces which are not at infinity of a desingularization, as in Section 2. Clearly, the space M of units is compact. Denoting by M_0 the interior of M , we have $M_0 = \Omega_0$, so Ω_0 is an open dense subset of M .

(We often replace \mathcal{J}'_{p_i} by \mathcal{J}_{p_i} in the definition of \mathcal{G} , where the gluing is always meant as above.)

Definition 4.1. The *layer potentials C^* -algebra* associated to a conical domain Ω is defined as the groupoid C^* -algebra $C^*(\mathcal{G})$, where \mathcal{G} is the boundary groupoid as in (13).

In the following results, we give a few properties of the boundary groupoid \mathcal{G} and of the layer potentials C^* -algebra.

Proposition 4.2. *Let \mathcal{G} be the groupoid (13) associated to a domain with conical points $\Omega \subset \mathbb{R}^n$. Let $\Omega^{(0)} = \{p_1, p_2, \dots, p_l\}$ be the set of conical points and $\Omega_0 = \partial\Omega \setminus \Omega^{(0)}$ be the smooth part of $\partial\Omega$. Then, \mathcal{G} is a Lie groupoid with units $M = \partial'\Sigma(\Omega)$ such that*

(1) $M_0 = \text{int}(M) = \Omega_0$ is an invariant subset and

$$\mathcal{G}_{M_0} \cong M_0 \times M_0.$$

(2) For each conical point $p \in \Omega^{(0)}$, $\{p\} \times \partial\omega_p$ and $\partial M = \bigcup_{p \in \Omega^{(0)}} \{p\} \times \partial\omega_p$ are invariant subsets and

$$\mathcal{G}|_{\partial M} = \prod_{i=1}^l (\partial\omega_i \times \partial\omega_i) \times (\mathbb{R}^+ \times \{p_i\})$$

(3) $A(\mathcal{G}) \cong {}^bTM$, the b -tangent bundle of M .

Proof. To show (1) and (2) note that if $m \in M_0$, then $\mathcal{G}_m = \partial\Omega \setminus \Omega^{(0)} = \Omega_0$ and that if $x \in \partial M$, then $\mathcal{G}_x \cong (0, \infty) \times \partial\omega_i$ for some $i \in \{1, \dots, l\}$.

To compute the Lie algebroid of \mathcal{G} , we see that if $\mathcal{G} = \mathcal{G}_1 \cup_\phi \mathcal{G}_2$, for Lie groupoids \mathcal{G}_i , $i = 1, 2$, and ϕ a diffeomorphism of an open set in \mathcal{G}_1 to an open set in \mathcal{G}_2 preserving the groupoid structure, then we can do a clutching construction on vector bundles to get that $A(\mathcal{G}) \cong A(\mathcal{G}_1) \cup_\phi A(\mathcal{G}_2)$. In our case, since $A(\Omega_0^2) = T(\Omega_0)$ and $A(\mathcal{J}'_i) = {}^bT([0, \infty) \times \partial\omega_i)$, according to Lemma 3.1, we have that

$$A(\mathcal{G}) = \left(\bigoplus_{i=1}^l {}^bT([0, \infty) \times \partial\omega_i) \right) \bigcup_{\varphi} T\Omega_0,$$

and therefore $\Gamma(A(\mathcal{G}))$ coincides with the vector fields on M that are tangent to $\{0\} \times \partial\omega_i$, $i = 1, \dots, l$, that is, tangent to ∂M . Hence, $A(\mathcal{G}) = {}^bTM$. \square

Recall that $C^*(\mathcal{G})$ was defined as the closure (in the full norm) of the class of order $-\infty$ pseudodifferential operators on \mathcal{G} (Definition 1.5).

Proposition 4.3. *Let \mathcal{G} be the boundary groupoid (13) associated to a domain with conical points $\Omega \subset \mathbb{R}^n$ as before. Then,*

(1) \mathcal{G} is amenable, i.e., $C^*(\mathcal{G}) \cong C_r^*(\mathcal{G})$.

(2) We have the following exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(\mathcal{G}) \longrightarrow \bigoplus_{i=1}^l \mathcal{C}_0(\mathbb{R}^+) \otimes \mathcal{K} \longrightarrow 0, \text{ if } n \geq 3,$$

and, if $n = 2$,

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(\mathcal{G}) \longrightarrow \bigoplus_{i=1}^l M_{k_i}(\mathcal{C}_0(\mathbb{R}^+)) \longrightarrow 0,$$

where k_i is the number of elements of $\partial\omega_i$ and l is the number of conical points.

Proof. To prove amenability, we first note that the pair groupoid is amenable (since the regular representations are essentially the only representations). Moreover, the disjoint union of amenable groupoids is amenable and the same holds for crossed products by amenable groups. Since \mathbb{R}^+ is abelian, hence amenable, we have that

$[0, \infty) \times (0, \infty) \times (\partial\omega_{p_i})^2$ is amenable. Since \mathcal{G} is given by the gluing of two amenable groupoids, it is amenable. (In fact, we only needed $\mathcal{G}_{M_0}, \mathcal{G}_{\partial M}$ amenable, see [48]).

As for the exact sequences, since ∂M is a closed, invariant submanifold of M , we have from (6) that

$$0 \longrightarrow C^*(\mathcal{G}|_{M \setminus \partial M}) \longrightarrow C^*(\mathcal{G}) \longrightarrow C^*(\mathcal{G}|_{\partial M}) \longrightarrow 0.$$

Assume first that $n \geq 3$. The C^* -algebra of the pair groupoid is isomorphic, through the vector representation, to the ideal of compact operators, and one can easily check that $C^*(\mathbb{R}^+ \times \{p_i\}) = \mathcal{C}_0(\mathbb{R})$. Hence, we obtain from (1) and (2) in Proposition 4.2,

$$C^*(\mathcal{G}_{M_0}) = \mathcal{K}, \quad \text{and} \quad C^*(\mathcal{G}|_{\partial M}) = \bigoplus_{i=1}^l \mathcal{K} \otimes \mathcal{C}_0(\mathbb{R}^+)$$

and the result follows in this case. If $n = 2$, then $\partial\omega_i$ are discrete sets, hence $C^*((\partial\omega_i)^2) \cong M_{k_i}(\mathbb{C})$, where k_i is the number of elements of $\partial\omega_i$. The exact sequence then follows in the same way. \square

Taking into account the remarks at the end of Section 3, we can characterize the class $\Psi^m(\mathcal{G})$ of pseudodifferential operators on our boundary groupoid:

Proposition 4.4. *Let \mathcal{G} be the boundary groupoid (13) with units M associated to a domain $\Omega \subset \mathbb{R}^n$ with conical points $\Omega^{(0)} = \{p_1, \dots, p_l\}$, $\Omega_0 = \Omega \setminus \Omega^{(0)}$ be the smooth part of $\partial\Omega$. Then*

- (1) $\Psi^m(\mathcal{G}_{M_0}) \cong \Psi^m(\Omega_0)$.
- (2) If $P \in \Psi^m(\mathcal{G}_{\partial M})$ then for each $p_i \in \Omega^{(0)}$, P defines a Mellin convolution operator on $\mathbb{R}^+ \times \partial\omega_i$.
- (3) $\Psi(\mathcal{G}) \supset \Psi({}^b\mathcal{G})$, the b -pseudodifferential operators on M .

Proof. From Proposition 4.2, we have $\mathcal{G}_{M_0} \cong (\Omega_0)^2$, so (1) follows from invariance. Moreover, for $P \in \Psi(\mathcal{G}_{\partial M})$, we have that on each invariant subset $\{p_i\} \times \partial\omega_i$, P defines a kernel $\kappa_i \in (\partial\omega_i)^2 \times (\mathbb{R}^+ \times \{p_i\})$, hence a Mellin convolution operator on $\mathbb{R}^+ \times \partial\omega_i$ with kernel $\tilde{\kappa}_i(r, s, x', y') := \kappa_i(r/s, x', y')$.

As for (3), note that for M as in (14), the b -groupoid becomes

$${}^b\mathcal{G} = M_0^2 \bigcup_{i,j} \mathbb{R}^+ \times (\partial_j\omega_i)^2,$$

where $\partial_j\omega_i$ denote the (open) connected components of $\partial\omega_i$, $i = 1, \dots, l$. Since

$$\mathbb{R}^+ \times (\partial_j\omega_i)^2 = d^{-1}(\partial_j\omega_i) \cap r^{-1}(\partial_j\omega_i) = \mathcal{G}_{\partial_j\omega_i}^{\partial_j\omega_i},$$

we have that ${}^b\mathcal{G}$ is an open subgroupoid of \mathcal{G} . The result follows. \square

We remark that it follows from the proof that if $\partial\omega_i$ is connected, for all $i = 1, \dots, l$, then $\mathcal{G} = {}^b\mathcal{G}$ (see also Remark 3.2).

In general, we also have that the b -indicial algebra $\Psi^0({}^b\mathcal{G}_{\partial M})$ is a $*$ -subalgebra of the indicial algebra $\Psi(\mathcal{G}_{\partial M})$. If $Q = (Q_x) \in \Psi^0({}^b\mathcal{G}_{\partial M})$, then for $x \in \partial_j\omega_i \subset \partial M$, we have $Q_x \in \Psi^0(\mathbb{R}^+ \times \partial_j\omega_i)$, so that it defines $P_x \in \Psi^0(\mathbb{R}^+ \times \partial\omega_i)$ by $P_x^{kk'} = Q_x$, if $k = k' = j$, and $P_x^{kk'} = 0$, otherwise, where $P_x^{kk'} : C_c^\infty(\mathbb{R}^+ \times \partial_{k'}\omega_i) \rightarrow C_c^\infty(\mathbb{R}^+ \times \partial_k\omega_i)$. For each $x \in \partial M$, we then identify Q_x with a diagonal entry in P_x , regarded as a matrix of operators. (Note that if $n = 2$, then we get a matrix of operators on

\mathbb{R}^+ .) The indicial algebra plays a key role in Fredholmness of operators on \mathcal{G} , as we shall see in Section 6.

To finish this subsection, we now give a result that shows that the boundary groupoid and the layer potentials C^* -algebra only depend, up to equivalence, on the number of singularities of the conical domain Ω . (Compare with a similar result for the b -groupoid of a manifold with corners [38]). Recall that two groupoids \mathcal{G} and \mathcal{G}' are *equivalent* if there exists a groupoid \mathcal{Z} containing \mathcal{G} and \mathcal{G}' as full subgroupoids, that is, if $\mathcal{G} = \mathcal{Z}_A^A := d_{\mathcal{Z}}^{-1}(A) \cap r_{\mathcal{Z}}^{-1}(A)$ for some subset $A \subset \mathcal{G}^0$ intersecting all orbits, and the same for \mathcal{G}' . (We also get \mathcal{G} and \mathcal{G}' equivalent to \mathcal{Z} .) In this case, it is an important result of Muhly, Renault and Williams that the C^* -algebras $C^*(\mathcal{G})$ and $C^*(\mathcal{G}')$ are Morita equivalent (see [43] for the definitions and result).

Theorem 4.5. *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be two conical domains with no cracks, and $\mathcal{G}, \mathcal{G}'$ be the boundary groupoids as in (13). If Ω and Ω' have the same number of true conical points then \mathcal{G} is equivalent to \mathcal{G}' . Hence, $C^*(\mathcal{G})$ is Morita equivalent to $C^*(\mathcal{G}')$.*

Proof. We construct a groupoid \mathcal{Z} that yields the equivalence. Let $p_i, p'_i, i = 1, \dots, l$ be the (true) conical points of Ω and Ω' , respectively. Define

$$Z := (\Omega \sqcup \Omega') / \sim$$

where we identify $p_i \sim p'_i$, for each $i = 1, \dots, l$. Then Z is also a conical domain with no cracks: we have $\partial Z \setminus \{[p_1], \dots, [p_l]\} = \Omega_0 \sqcup \Omega'_0$, with Ω_0 and Ω'_0 the smooth parts of $\partial\Omega$ and $\partial\Omega'$, respectively, hence it is smooth. If $x = [p_i]$, then $\omega_{p_i} \sqcup \omega_{p'_i}$ is a basis for a local cone at x (note that $\omega_{p_i} \sqcup \omega_{p'_i} \neq S^{n-1}$ since p_i and p'_i are true vertices, and by definition $\overline{\omega_{p_i} \sqcup \omega_{p'_i}} = \overline{\omega_{p_i}} \sqcup \overline{\omega_{p'_i}}$, so Z has no cracks).

Let \mathcal{Z} be the boundary groupoid (13) associated to Z . It follows from (14) that the units of \mathcal{Z} coincide with $M \sqcup M'$, where M, M' are units of $\mathcal{G}, \mathcal{G}'$. Moreover, we have $\mathcal{G} = \mathcal{Z}_M^M$ and we check that M intersects all orbits. Over the interior of $M \sqcup M'$, the orbits are given by $\Omega_0 \cup \Omega'_0$, which intersects M . Over the boundary, the orbit at $([p_i], x)$, with $x \in \partial\omega_{p_i} \cup \partial\omega_{p'_i}$ is $(\omega_{p_i} \sqcup \omega_{p'_i})$, which also intersects M .

Hence, \mathcal{G} is a full subgroupoid of \mathcal{Z} . Since the same holds for \mathcal{G}' with respect to M' , we conclude that \mathcal{G} and \mathcal{G}' are equivalent. The last assertion follows from the main result in [43]. \square

4.2. Groupoid construction for polygonal domains with ramified cracks.

In this subsection, we consider a polygonal domain $\Omega \subset \mathbb{R}^2$. We allow Ω to have ramified cracks (see the definition below). To any such domain, we associate a generalized conical domain, the so-called *unfolded domain* Ω^u , with no cracks, and apply the groupoid construction of the previous section.

We first give a precise definition of cracks. We distinguish between different kinds of crack points, even though we will basically need the distinction between smooth and singular crack points. The case of conical crack points is also considered, where we need to distinguish between the crack part and the non-crack parts of the conical neighborhood.

Definition 4.6. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain as in Definition 2.1. Let $x \in \partial\Omega$. Suppose that there exists a neighborhood V_x of x in Ω , an open subset

$\omega_x \subset S^1$, $\omega_x \neq S^1$, and a diffeomorphism $\phi_x : V_x \rightarrow B^2$ such that in polar coordinates (r, θ) , we have

$$\phi_x(V_x \cap \Omega) = \{(r, \theta), r \in (0, 1), \theta \in \omega_x\}.$$

Then x is a *crack point* if $\partial\omega_x \neq \overline{\partial\omega_x}$. Moreover:

- (1) x is an *inner crack point* if $\overline{\omega_x} = S^1$. If ω_x is isomorphic to $S_+^1 \sqcup S_+^1$, where S_+^1 is the hemisphere, we call x a *smooth crack point*.
- (2) x is an *outer crack point* if $\overline{\omega_x} = S_+^1$.
- (3) otherwise, x is a *conical crack point*.

Remark 4.7. In the above definition, we shall always assume that crack curves intersect transversally.

Remark 4.8. By Definition 4.6, any non-smooth point on the boundary $\partial\Omega$ can be classified to be one of true conical non-crack points, inner, outer, and conical crack points. Moreover, it is easy to check that the set of smooth crack points is given by a union of smooth curves, the actual cracks, with boundary given by inner, outer or possibly conical crack points. In higher dimensions, even if Definition 4.6 would still apply, a crack would be a codimension 1 submanifold and its boundary would be a codimension 2 submanifold, corresponding to non-smooth crack points of $\partial\Omega$. Hence Ω would not be a conical domain, in the sense that the set of singular points is not discrete (and finite). In this case, the boundary points of the crack behave like edge points, rather than conical points, and Ω would be the interior of a manifold with corners.

To each crack point x , we now associate a *ramification number*, which will turn out to be very useful to define the unfolded domain as it gives us the number of different ways we can approach the boundary close to x . It is defined as follows.

- (1) If x is a inner or outer crack point, we let k be the number of connected components of ω_x (so ω_x is isomorphic to the disjoint union of k copies of $(0, 2\pi/k)$ or $(0, \pi/k)$, for inner and outer crack points, respectively). Then we say that x is a crack point with k *ramifications*. (In particular, if x is a smooth crack point, then its ramification number is 2).
- (2) If x is a conical crack point, then we split $\omega_x = \omega'_x \cup \omega''_x$, where ω'_x and ω''_x are disjoint and given by unions of connected components of ω_x such that $I_j \subset \omega'$ if and only if $\overline{I_j} \cap \overline{I_i} = \emptyset$ for all $i \neq j$, where I_j are the connected components of ω , so $\partial\omega' = \overline{\partial\omega'_x}$ and $\partial\omega'' \neq \overline{\partial\omega''_x}$. Note that ω''_x is non-empty (otherwise, we get a true conical non-crack point.) Let k be the number of connected components of ω''_x . If $\omega'_x = \emptyset$, we say that x is a conical crack point with k *ramifications*. If $\omega'_x \neq \emptyset$, we say that x is a conical crack point with $k + 1$ *ramifications*.

If x is a conical crack point, we always refer to ω'_x and ω''_x as the no-crack and the crack part of ω_x , respectively. For instance, if $\omega_x = (0, \pi/4) \cup (\pi/4, \pi/2) \cup (\pi, 5\pi/4) \cup (3\pi/2, 7\pi/4)$, then $\omega'_x = (\pi, 5\pi/4) \cup (3\pi/2, 7\pi/4)$, $\omega''_x = (0, \pi/4) \cup (\pi/4, \pi/2)$ and its ramification number is 3. Also, a point with ramification 1 means that it is an end point of some crack curve. We refer to conical/outer/inner crack points with $k \neq 2$, as *singular crack points*, as they are true conical points in the sense of Definition 2.1 (that is, singularities of $\partial\Omega$).

For the remainder of this subsection, we denote by $\Omega^{(0)} = \{p_1, \dots, p_l\}$ the set of true conical non-crack points. (Note that here $\Omega^{(0)}$ does not include all conical

points in the sense of Definition of 2.1.) Also, we let $\mathcal{C} := \{c_1, \dots, c_m\}$ be the set of singular crack points, with $c_1, \dots, c_{m'}$ the conical crack points for which $\omega'_{c_j} \neq \emptyset$.

Now, if x is a smooth boundary point, then the intersection of a small neighborhood of x and $\partial\Omega$ is a smooth curve γ . We then have two possibilities: either Ω lies on one side of γ , or Ω lies in both sides of γ (the latter corresponds to a smooth crack point). Hence, we have one inward normal unit vector in the first case, and two inward normal unit vectors in the second case.

The most important idea behind the construction of the unfolded domain is that each smooth crack point x should be covered by two points, which correspond to the two sides of the crack and the two possible non-tangential limits at x of functions defined on Ω . (See [26, 29] for details.) The point is to cover each smooth crack curve by two parallel smooth curves, so that we distinguish the two normal directions from which we approach the boundary. When cracks ramify, we need further to distinguish from which side we approach the point of ramification. In general, if a crack point has ramification k , then we cover this point by k points.

Following [29], we define the *unfolded boundary* $\partial^u\Omega$ as the set of inward pointing unit normal vectors to the smooth part Ω_0 of $\partial\Omega$. We have a canonical projection $\kappa : \partial^u\Omega \rightarrow \Omega_0$, which is two-to-one over smooth crack points and one-to-one over the non-crack smooth points. We then define the *unfolded domain* as

$$(15) \quad \Omega^u := \Omega \cup \partial^u\Omega.$$

We can think of Ω^u as a (generalized) polygonal domain, without cracks. (The angles at some points may be 2π , i.e., some two adjacent edges may be parallel.) The boundary of Ω^u is given by $\partial^u\Omega$ together with the (true) non-crack vertices of Ω , together with k -covers of crack points with ramification k . Note that the smooth part of $\partial(\Omega^u)$ is given by $\partial^u\Omega$, that is, by the union of the smooth part of $\partial\Omega$ and the 2-covers of the smooth crack curves.

We will let, from now on, $\mathcal{C}^u := \{c_{ji} \mid c_{ji} \text{ covers } c_j \in \mathcal{C}, i = 1, \dots, k_{c_j}\} \subset \partial(\Omega^u)$, where k_{c_j} is the number of connected components of ω_{c_j} , if c_j is not a conical crack point (that is, the ramification number), or the number of connected components of ω''_{c_j} , if c_j is a conical crack point. In this case, if $\omega'_{c_j} \neq \emptyset$, we regard the cover over c_j as a non-crack point c_{j0} together with a k_{c_j} -cover.

With the notation as above, we see that the set of (true) vertices of Ω^u is given by

$$(16) \quad V^u = \Omega^{(0)} \cup \mathcal{C}^u \cup \{c_{j0}\}_{j=1, \dots, m'}.$$

Moreover, we have that $\Omega^u \setminus V^u$ is given by a disjoint union of smooth curves, as in the no-cracks case.

For $x \in V^u$, we denote by ω_x^u the basis of the cone at x in Ω^u . If $c \in \mathcal{C} \subset \partial\Omega$ is a singular outer or inner crack point and the basis of the cone at c (in Ω) is written as a union of open intervals $\omega_c \cong \bigcup_{h=1}^{k_c} I_h$, then for each $c_h \in \mathcal{C}^u \subset \partial(\Omega^u)$ in a k_c -cover of c , we have $\omega_{c_h}^u = I_h$ is connected, and there is a neighborhood V_{c_h} such that $V_{c_h} \cap \Omega^u \cong (0, 1) \times \omega_{c_h}^u$. If c is a conical crack point, then the same as above happens, replacing ω_c by the crack part ω_c'' . At the 'non-crack' point c_0 , we have $\omega_{c_0}^u = \omega_c'$ (possibly disconnected).

Now we desingularize Ω^u and follow the same construction as in Subsection 4.1 for Ω^u . We obtain a groupoid \mathcal{G}^u with the unit space isomorphic to $\partial'\Sigma(\Omega^u)$, i.e., to a desingularization of $\partial\Omega^u$. Denote by $\mathcal{H} = [0, 1) \rtimes (0, 1)$ the transformation groupoid. From (13), we define the *boundary groupoid associated to a conical*

domain with ramified cracks as

$$\begin{aligned}\mathcal{G}^u &:= \left(\coprod_{x \in V^u} \mathcal{H} \times (\partial\omega_x^u)^2 \right) \bigcup_{\varphi} (\partial^u\Omega)^2 \\ &= \left(\coprod_{p_i \in \Omega^{(0)}} \mathcal{H} \times (\partial\omega_{p_i})^2 \bigcup_{j=1}^{m'} \coprod \mathcal{H} \times (\partial\omega'_{c_j})^2 \bigcup \coprod_{c_{jh} \in \mathcal{C}^u} \mathcal{H} \times (\partial I_{c_j}^h)^2 \right) \\ &\quad \bigcup_{\varphi} (\partial^u\Omega)^2,\end{aligned}$$

where $I_{c_j}^h$ is the h -th connected component of ω_{c_j} , respectively, of ω'_{c_j} , if c_j is non-conical, respectively, c_j is a conical crack point, and $\varphi = (\varphi_x)_{x \in V^u}$. Noting that $\partial\omega_x$ is a discrete set, with ω_x connected on covers of crack points, and denoting by \mathcal{P}_k the pair groupoid of a discrete set with k elements, we have (writing $(\coprod A)^\alpha$ for the disjoint union of α copies of A):

$$(17) \quad \mathcal{G}^u = \left(\coprod_{p_i \in \Omega^{(0)}} \mathcal{H} \times \mathcal{P}_{2k_{p_i}} \bigcup_{j=1, \dots, m'} \coprod \mathcal{H} \times \mathcal{P}_{2k'_{c_j}} \bigcup \left(\coprod \mathcal{H} \times \mathcal{P}_2 \right)^\alpha \right) \bigcup_{\varphi} (\partial^u\Omega)^2,$$

where k'_{c_j} is the number of connected components of ω'_{c_j} and $\alpha := k_{c_1} + \dots + k_{c_m}$ is a total ramification number of Ω . The space of units of \mathcal{G}^u is given by

$$\begin{aligned}(18) \quad M &:= \left(\coprod_{p_i \in \Omega^{(0)}} [0, 1] \times \partial\omega_{p_i} \bigcup \coprod_{j=1}^{m'} [0, 1] \times \partial\omega'_{c_j} \bigcup \coprod_{c_{ji} \in \mathcal{C}^u} [0, 1] \times \partial\omega_{c_{ji}}^u \right) \bigcup_{\varphi} \partial^u\Omega \\ &\cong \left(\coprod_{i=1}^l \left(\coprod [0, 1] \right)^{2k_{p_i}} \bigcup \coprod_{j=1}^{m'} \left(\coprod [0, 1] \right)^{2k'_{c_j}} \bigcup \left(\coprod [0, 1] \right)^{2\alpha} \right) \bigcup_{\varphi} \partial^u\Omega,\end{aligned}$$

which is diffeomorphic to a desingularization of $\partial(\Omega^u)$, as in Section 2. The unfolded boundary $\partial^u\Omega$ is an open dense set of M and

$$\partial M = \bigcup_{x \in V^u} \{x\} \times \partial\omega_x^u = \bigcup_{p \in \Omega^{(0)}} \{p\} \times \partial\omega_p \bigcup_{c \in \mathcal{C}^u} \{c\} \times \partial\omega_c^u \bigcup_{j=1}^{m'} \{c_j\} \times \partial\omega'_{c_j}.$$

Definition 4.9. The *layer potentials C^* -algebra* associated to a polygonal domain Ω with ramified cracks is defined as the groupoid C^* -algebra $C^*(\mathcal{G}^u)$, where \mathcal{G}^u is the unfolded boundary groupoid as in (17).

We summarize below the main properties of \mathcal{G}^u and its C^* -algebra (see Propositions 4.2 and 4.3). As above, we let $\Omega^{(0)} = \{p_1, p_2, \dots, p_l\}$ be the set of conical, non-crack points of Ω , $\mathcal{C} = \{c_1, \dots, c_m\}$ be the set of singular crack points, with $c_1, \dots, c_{m'}$ the conical crack points with not empty no-cracks part, and $\mathcal{C}^u := \{c_{ji} \mid c_{ji} \text{ covers } c_j \in \mathcal{C}, i = 1, \dots, k_{c_j}\}$.

Proposition 4.10. *Let \mathcal{G}^u be the groupoid (17) associated to a polygonal domain with cracks $\Omega \subset \mathbb{R}^2$. Denote by $\partial^u\Omega$ the unfolded boundary. Then, \mathcal{G}^u is a Lie groupoid with units M such that*

- (1) $M_0 = \text{int}(M) = \partial^u\Omega$ is an invariant subset and $\mathcal{G}_{M_0}^u \cong M_0 \times M_0$.

- (2) $\{x\} \times \partial\omega_x^u \subset \partial M$ is an invariant subset, for $x \in V^u$, and ∂M is an invariant subset with

$$\begin{aligned} \mathcal{G}_{\partial M}^u &= \prod_{i=1}^l (\partial\omega_{p_i})^2 \times (\mathbb{R}^+ \times \{p_i\}) \cup \prod_{j=1}^{m'} (\partial\omega'_{c_j})^2 \times (\mathbb{R}^+ \times \{c_j\}) \\ &\quad \cup \prod_{c_{jh} \in \mathcal{C}^u} (\partial I_{c_j}^h)^2 \times (\mathbb{R}^+ \times \{c_{jh}\}) \end{aligned}$$

where $I_{c_j}^h$ is the h -th connected component of ω_{c_j} , if $j > m'$, or of ω''_{c_j} , if $m \leq m'$.

- (3) $A(\mathcal{G}^u) \cong {}^b TM$, the b -tangent bundle of M .
 (4) \mathcal{G}^u is amenable, i.e., $C^*(\mathcal{G}^u) \cong C_r^*(\mathcal{G}^u)$.
 (5) We have the following exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(\mathcal{G}^u) \longrightarrow \mathcal{A} \longrightarrow 0,$$

$$\text{with } \mathcal{A} := \left(\bigoplus_{i=1}^l M_{k_{p_i}}(\mathcal{C}_0(\mathbb{R}^+)) \right) \oplus \left(\bigoplus_{j=1}^{m'} M_{k'_{c_j}}(\mathcal{C}_0(\mathbb{R}^+)) \right) \oplus \left(\bigoplus_{\alpha \text{ times}} M_2(\mathcal{C}_0(\mathbb{R}^+)) \right),$$

where k_{p_i}, k'_{c_j} denote the number of elements of $\partial\omega_{p_i}$ and $\partial\omega'_{c_j}$, respectively, and $\alpha = k_{c_1} + \cdots + k_{c_m}$ is the total ramification number of Ω (see Equation 17).

We also have an analogue of Theorem 4.5 for polygonal domains with ramified cracks. Recall from (16) that the number of true conical points of Ω^u is given by $\sharp(V^u) = l + m' + \alpha$ with l the number of true conical non-crack points of Ω , m' the number of conical crack points with a not empty no-crack part and $\alpha := k_1 + \cdots + k_m$ can be regarded as the total ramification number of Ω (with m is the number of all singular crack points). We then have:

Proposition 4.11. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two polygonal domains, possibly with cracks, and $\mathcal{G}_1^u, \mathcal{G}_2^u$ be the unfolded boundary groupoids as in (17). If, using the notation above, $l_1 + m'_1 + \alpha_1 = l_2 + m'_2 + \alpha_2$, then \mathcal{G}_1^u is equivalent to \mathcal{G}_2^u and $C^*(\mathcal{G}_1^u)$ is Morita equivalent to $C^*(\mathcal{G}_2^u)$*

5. K -THEORY OF THE LAYER POTENTIALS C^* -ALGEBRA

In this section, we compute the K -theory groups of the layer potentials C^* -algebra of the boundary groupoid of a conical domain and of the associated indicial algebras. We obtain, in particular, that the groupoid K -theory is the same for domains with the same number of singularities. We refer to [7, 49] for the general concepts and results regarding K -theory for C^* -algebras. See also [22, 32, 38, 41, 44] for some computations of the K -groups of C^* -algebras associated to singular domains.

We consider first the case of a straight cone $\Omega = \mathbb{R}^+ \omega \subset \mathbb{R}^n$, with $\omega \subset S^{n-1}$ and let $\mathcal{J} = \mathcal{H} \times (\partial\omega)^2$ as in Section 3, where $\mathcal{H} = [0, \infty) \rtimes \mathbb{R}^+$. Then, from Lemma 3.3, if $n \geq 3$,

$$C^*(\mathcal{J}) \cong C^*(\mathcal{H}) \otimes \mathcal{K} \cong (\mathcal{C}_0([0, \infty)) \rtimes \mathbb{R}^+) \otimes \mathcal{K}.$$

and, if $n = 2$,

$$C^*(\mathcal{J}) \cong C^*(\mathcal{H}) \otimes M_k(\mathbb{C}) = M_k(C^*(\mathcal{H})),$$

where k is the number of elements of $\partial\omega$. Since $K_j(C^*(\mathcal{H}) \otimes \mathcal{K}) = K_j(M_k(C^*(\mathcal{H}))) = K_j(C^*(\mathcal{H}))$, $j = 0, 1$, and since, by the Connes' Thom isomorphism [7]

$$K_j(\mathcal{C}_0([0, \infty)) \rtimes \mathbb{R}^+) = K_{1-j}(\mathcal{C}_0([0, \infty))) = 0, \quad j = 0, 1$$

(see [49]) we get that

$$(19) \quad K_*(C^*(\mathcal{J})) = 0$$

for the layer potentials C^* -algebra of a straight cone.

Now we consider the general case. We start with computing the K -groups of the indicial algebra, i.e., at the boundary. We assume first that $n \geq 3$. As in Definition 2.1, we denote by $\Omega \subset \mathbb{R}^n$ a domain with distinct conical points $\{p_1, p_2, \dots, p_l\}$, with no cracks. Also denote by $\omega_i \subset S^{n-1}$ the basis of the cone that corresponds to p_i . (Note that ω_i may be disconnected). Following the notations in Section 4, we have

$$\mathcal{G}|_{\partial M} = \prod_{i=1}^l (\partial\omega_i \times \partial\omega_i) \times (\mathbb{R}^+ \times \{p_i\})$$

with unit space ∂M . Since $C^*((\partial\omega_i)^2) = \mathcal{K}$ and $C^*(\mathbb{R}^+ \times \{p_i\}) = \mathcal{C}_0(\mathbb{R})$, we obtain

$$C^*(\mathcal{G}|_{\partial M}) = \bigoplus_{i=1}^l \mathcal{K} \otimes \mathcal{C}_0(\mathbb{R}).$$

Consider next the case $n = 2$, where we allow Ω to have cracks. Let V^u be the set of vertices of the unfolded domain Ω^u , as in the previous section (so $x \in V^u$ if x is a non-crack conical point of Ω or x is in a cover of a singular crack point). Denote by ω_x^u the basis of the cone that corresponds to a vertex x in Ω^u . In this case, $\partial\omega_x^u$ is a discrete set. Let k_x be the number of points in $\partial\omega_x^u$ (k_x is even.) We have $C^*(\partial\omega_x^u \times \partial\omega_x^u) = C^*(\mathcal{P}_{k_x}) = M_{k_x}(\mathbb{C})$. Hence, we obtain

$$C^*(\mathcal{G}|_{\partial M}) = \bigoplus_{x \in V^u} M_{k_x}(\mathbb{C}) \otimes \mathcal{C}_0(\mathbb{R}) = \bigoplus_{x \in V^u} M_{k_x}(\mathcal{C}_0(\mathbb{R})).$$

In any case, since $K_j(\mathcal{K} \otimes \mathcal{C}_0(\mathbb{R})) = K_j(M_k(\mathbb{C}) \otimes \mathcal{C}_0(\mathbb{R})) = K_j(\mathcal{C}_0(\mathbb{R}))$ for $j = 1, 2$, and $K_0(\mathcal{C}_0(\mathbb{R})) = 0$, $K_1(\mathcal{C}_0(\mathbb{R})) = \mathbb{Z}$ (see [49]), we have

Lemma 5.1. *Let l denote the number of conical points in Ω , $n \geq 3$ or in Ω^u , for $n = 2$ (in this case, including crack points). Then*

$$K_0(C^*(\mathcal{G}|_{\partial M})) = 0 \quad \text{and} \quad K_1(C^*(\mathcal{G}|_{\partial M})) = \bigoplus_{i=1}^l \mathbb{Z} = \mathbb{Z}^l.$$

Since there is an interaction between connected components at the groupoid level, the K -groups do not depend on the (number of) connected components of ω_i and $\partial\omega_i$. On the other hand, it is clear from the computations above that the number of conical points plays an essential role for the K -theory of $C^*(\mathcal{G}|_{\partial M})$. This fact should be expected from Theorem 4.5, since Morita equivalent C^* -algebras have isomorphic K -groups. Hence:

Corollary 5.2. *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be two conical domains, and let $\mathcal{G}, \mathcal{G}'$ be the boundary groupoids as in (13). Assume Ω and Ω' have the same number of true conical points. Then*

$$K_j(C^*(\mathcal{G})) \cong K_j(C^*(\mathcal{G}')), \quad j = 0, 1.$$

A similar result holds also for domains with cracks, replacing Ω by Ω^u , that is, ‘number of true conical points’ by ‘ $l + m' + \alpha$ ’ as in Corollary 4.11.

This result allows us to replace, at the level of K -theory, a conical domain by any other with the same number of true vertices.

Remark 5.3. The K -theory of the b -groupoid ${}^b\mathcal{G}$ with units M was computed in [32, 38], in the more general context of manifolds with corners. If M has a smooth boundary, one gets $K_0(C^*({}^b\mathcal{G})) = 0$, $K_1(C^*({}^b\mathcal{G})) = \mathbb{Z}^{p-1}$, where p is the number of connected components of ∂M . As we noted, if Ω is a conical domain such that $\partial\omega_i$ connected for all conical points (so $n \geq 3$), we have $\mathcal{G} = {}^b\mathcal{G}$, where \mathcal{G} is the boundary groupoid associated to Ω , and in that case the number of connected components of ∂M is exactly the number of conical points. We therefore obtain, for such Ω ,

$$(20) \quad K_0(C^*(\mathcal{G})) = 0, \quad K_1(C^*(\mathcal{G})) = \mathbb{Z}^{l-1},$$

where l is the number of true conical points. Now, given *any* conical domain $\Omega \subset \mathbb{R}^n$, for $n \geq 3$, with no cracks, we can use Corollary 5.2 and the fact that one can always construct $\Omega' \subset \mathbb{R}^n$ with the same (number of) true conical points of Ω and such that the basis $\omega_i \subset S^{n-1}$ of the cone at the vertices has $\partial\omega_i$ connected, to conclude that (20) also holds in the general case, if the dimension is greater than two.

We saw that if $n \geq 3$, then the K -groups of the layer potential C^* -algebra can be computed from the K -groups of the b -groupoid C^* -algebra, using Corollary 5.2. We will give a simple, direct proof that (20) holds in general, that works as well for $n = 2$, and does not use groupoid equivalence, nor the computations for the b -groupoid. The main point is to use functoriality to reduce to the case of a straight cone and make use of the fact that in this case the K -groups of the boundary groupoid are trivial.

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a conical domain with no cracks and \mathcal{G} be the boundary groupoid as in (13). Then*

$$K_0(C^*(\mathcal{G})) = 0, \quad K_1(C^*(\mathcal{G})) = \mathbb{Z}^{l-1},$$

where l is the number of true conical points of Ω .

Proof. By Proposition 4.3, we have the following six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \xrightarrow{i_*} & K_0(C^*(\mathcal{G})) & \xrightarrow{q_*} & K_0(C^*(\mathcal{G}|_{\partial M})) \\ \delta \uparrow & & & & \downarrow \\ K_1(C^*(\mathcal{G}|_{\partial M})) & \xleftarrow{q_*} & K_1(C^*(\mathcal{G})) & \xleftarrow{i_*} & K_1(\mathcal{K}), \end{array}$$

where i_* is induced by the inclusion and q_* by the restriction map, and δ is the index map. Therefore, since $K_0(\mathcal{K}) = \mathbb{Z}$, $K_1(\mathcal{K}) = 0$ (see [49]), we obtain from Lemma 5.1

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{i_*} & K_0(C^*(\mathcal{G})) & \xrightarrow{q_*} & 0 \\ \delta \uparrow & & & & \downarrow \\ \mathbb{Z}^l & \xleftarrow{q_*} & K_1(C^*(\mathcal{G})) & \xleftarrow{i_*} & 0. \end{array}$$

We shall show that δ is surjective, which yields $K_0(C^*(\mathcal{G})) = 0$ (since in that case i_* is surjective and the zero map), and that $\ker \delta \cong \mathbb{Z}^{l-1}$, which proves the result, as $K_1(C^*(\mathcal{G})) \cong \text{Im}(q_*) = \ker \delta$.

Now let $V_l \cong (0, 1)\omega_l$ be a conical neighborhood of the conical point $p_l \in \Omega^{(0)}$. Then V_l is open in Ω and $\mathcal{G}_{V_l} = \mathcal{H} \times (\partial\omega_l)^2$ is an open subgroupoid of $\mathcal{G} = \mathcal{G}_\Omega$, with units $[0, 1] \times \partial\omega_l \subset M$ open as well, and $\partial([0, 1] \times \partial\omega_l) = \partial M \cap [0, 1] \times \partial\omega_l$. Hence, there is an induced morphism of C^* -algebras $i_l : C^*(\mathcal{G}_{V_l}) \rightarrow C^*(\mathcal{G})$, obtained by extending compactly supported functions on \mathcal{G}_{V_l} to \mathcal{G} by 0. At the interior, that is, for pair groupoids, denoting by $M_1 := (0, 1) \times \partial\omega_l$ we get a map

$$k_l : C^*(M_1 \times M_1) \cong \mathcal{K}(L^2(M_1)) \rightarrow C^*(M_0 \times M_0) \cong \mathcal{K}(L^2(M_0)),$$

which is injective but not surjective, as its image is given by the operators with smooth kernel supported in $M_1 \times M_1$. If we let $j_l : C^*((\partial\omega_l)^2 \times \mathbb{R}^+) \rightarrow C^*(\mathcal{G}_{\partial M})$ be the inclusion, we have now a commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^*((0, 1) \times \partial\omega_l)^2 = \mathcal{K} & \longrightarrow & C^*(\mathcal{G}_{V_l}) & \longrightarrow & C^*((\partial\omega_l)^2 \times \mathbb{R}^+) \longrightarrow 0 \\ & & \downarrow k_l & & \downarrow i_l & & \downarrow j_l \\ 0 & \longrightarrow & C^*(M_0 \times M_0) = \mathcal{K} & \longrightarrow & C^*(\mathcal{G}) & \longrightarrow & C^*(\mathcal{G}_{\partial M}) \longrightarrow 0, \end{array}$$

By functoriality of the index map, it follows that we have moreover a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} = K_1(C^*((\partial\omega_l)^2 \times \mathbb{R}^+)) & \xrightarrow{\delta'} & K_0(\mathcal{K}(L^2(M_1))) = \mathbb{Z} \\ (j_l)_* \downarrow & & (k_l)_* \downarrow \cong \\ \mathbb{Z}^l = K_1(C^*(\mathcal{G}_{\partial M})) & \xrightarrow{\delta} & K_0(\mathcal{K}(L^2(M_0))) = \mathbb{Z}, \end{array}$$

where $(j_l)_*(z) = (0, \dots, 0, z) \in \mathbb{Z}^l$ and δ' is the index map for the straight cone V_l . We claim that $(k_l)_* : K_0(\mathcal{K}(L^2(M_1))) = \mathbb{Z} \rightarrow \mathbb{Z} = K_0(\mathcal{K}(L^2(M_0)))$ is an isomorphism. This follows since for any $n \in \mathbb{N}$ we can find a projection in $\mathcal{K}(L^2(M_0))$ with kernel compactly supported in M_1 and $\dim p(L^2(M_0)) = \dim p(L^2(M_1)) = n$, so $(k_l)_*$ is surjective. (Recall that the identification of the K_0 -group of the compact operators with \mathbb{Z} is such that the equivalence class of a compact projection gets mapped to the dimension of its image, see for instance [49].)

Now, as we saw in the beginning of this section, $K_*(\mathcal{G}_{V_l}) = 0$ for a straight cone, and therefore the index map δ' is an isomorphism. Hence, $\delta \circ (j_l)_*$ is also an isomorphism, so it follows straightaway that δ is surjective and $K_0(C^*(\mathcal{G})) = 0$.

For K_1 , note that we have a split exact sequence

$$0 \longrightarrow \ker \delta \longrightarrow K_1(C^*(\mathcal{G}_{\partial M})) \xrightarrow{\delta} K_0(\mathcal{K}) \longrightarrow 0,$$

where the splitting is given by $\phi : K_0(\mathcal{K}) \rightarrow K_1(C^*(\mathcal{G}_{\partial M}))$, $\phi := (j_l)_*(\delta')^{-1}(k_l)_*^{-1}$, hence $\mathbb{Z}^l = K_1(C^*(\mathcal{G}_{\partial M})) \cong \ker \delta \oplus K_0(\mathcal{K})$, so that $\ker \delta \cong \mathbb{Z}^{l-1} = K_1(C^*(\mathcal{G}))$. \square

It follows immediately that:

Corollary 5.5. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain possibly with cracks and \mathcal{G}^u be the unfolded boundary groupoid as in (17). Then*

$$K_0(C^*(\mathcal{G}^u)) = 0, \quad K_1(C^*(\mathcal{G}^u)) = \mathbb{Z}^{l+m'+\alpha-1},$$

where l is the number of true conical no-crack points of Ω , m' is the number of conical crack points with a not empty no-crack part and α is the total ramification number of Ω .

The group $K_1(C^*(\mathcal{G}))$ is the recipient of the so-called analytic index for operators on groupoids [40] given by the connecting map in the exact sequence (5) associated to the principal symbol of pseudodifferential operators on \mathcal{G}

$$\delta : K_0(\mathcal{C}_0(S^*A)) = K^0(S^*A) \rightarrow K_1(C^*(\mathcal{G})) = \mathbb{Z}^{l-1},$$

where S^*A is the sphere Lie algebroid of \mathcal{G} . If $P \in \overline{\Psi^0(\mathcal{G})}$ is elliptic, then its symbol defines a class $[\sigma(P)] \in K^0(S^*A)$ and the analytic index of P is defined as $an - ind(P) := \delta[\sigma(P)]$. (The analogue of δ in the smooth case maps $K^0(S^*M) \rightarrow K_1(\mathcal{K}) = \mathbb{Z}$ and computes the Fredholm index.)

In applications, we are often more interested in studying the actual Fredholmness of (a representation of) $P \in \Psi^\infty(\mathcal{G})$, which is what we shall do next.

6. FREDHOLM CRITERION

In this section, we obtain a Fredholm criterion for pseudodifferential operators the boundary groupoid \mathcal{G} associated to a conical domain, possibly with cracks in two dimensions, as in (13) and (17).

Recall that $\Psi^m(\mathcal{G})$ was defined in Section 1 as the space of smooth, left-invariant, uniformly supported families $(P_x)_{x \in M}$, with $P_x \in \Psi^m(\mathcal{G}_x)$ and that there is a well-defined symbol map $\sigma_0 : \overline{\Psi^0(\mathcal{G})} \rightarrow \mathcal{C}(S^*A)$, where $A = A(\mathcal{G})$. The vector representation π of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(M)$ is uniquely determined by the equation $(\pi(P)f) \circ r := P(f \circ r)$, where $f \in \mathcal{C}_c^\infty(M)$ and $P = (P_x) \in \Psi^m(\mathcal{G})$. Let also π_x , $x \in M$ be the regular representation of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(\mathcal{G}_x)$ with $\pi_x(P) = P_x$.

To obtain our main results, we will use the spectral properties of $\Psi^m(\mathcal{G})$ obtained in [21] (see also [20]). We first consider the invariant set ∂M . It follows from the exact sequence (7), using once more the fact that the C^* -algebra of the pair groupoid is isomorphic to the compact operators, that we have

$$(21) \quad 0 \longrightarrow \mathcal{K} \xrightarrow{\pi} \overline{\Psi^0(\mathcal{G})} \xrightarrow{\mathcal{R}_{\partial M} \oplus \sigma_0} \overline{\Psi^0(\mathcal{G}_{\partial M})} \times_{\mathcal{C}_0(S^*A_{\partial M})} \mathcal{C}_0(S^*A) \longrightarrow 0,$$

where $\mathcal{R}_{\partial M} : \Psi^m(\mathcal{G}) \rightarrow \Psi^m(\mathcal{G}_{\partial M})$ is the restriction map. Hence, $P \in \overline{\Psi^0(\mathcal{G})}$ has a representation as a Fredholm operator on $L^2(M)$ if, and only if, $(\mathcal{R}_{\partial M} \oplus \sigma_0)(P)$ is invertible, that is, if P is elliptic and the restriction to the indicial algebra $\mathcal{R}_{\partial M}(P)$ is invertible. It also follows that

$$\mathcal{K} = \ker(\mathcal{R}_{\partial M} \oplus \sigma_0) = (\ker \mathcal{R}_{\partial M}) \cap C^*(\mathcal{G}).$$

Moreover, we have (see Theorem 9.3 in [21], and also Theorem 4 in [20]):

Theorem 6.1. *Let \mathcal{G} be a Lie groupoid with units M and M_0 be an invariant open dense subset of M such that $\mathcal{G}|_{M_0} \cong M_0 \times M_0$. Assume that the restriction of \mathcal{G} to $M \setminus M_0$ is amenable, and that the vector representation π is injective. Let $P \in \Psi^m(\mathcal{G})$ or $P \in \overline{\Psi^0(\mathcal{G}_{\partial M})}$. Then*

- (1) $\pi(P) : H^m(M) \rightarrow L^2(M)$ is Fredholm if, and only if, P is elliptic and $\pi_x(P) := P_x : H^m(\mathcal{G}_x) \rightarrow L^2(\mathcal{G}_x)$ is invertible, for all $x \notin M_0$.
- (2) $\pi(P) : H^m(M) \rightarrow L^2(M)$ is compact if, and only if, $\sigma_0(P) = 0$ and $\pi_x(P) = 0$, for all $x \notin M_0$.

We will use this theorem for the boundary groupoid \mathcal{G} with the space of units M as in (13) and for the unfolded groupoid \mathcal{G}^u , on polygonal domains with cracks (17). It follows from Proposition 13 that $M_0 \simeq \Omega_0$ is an invariant open dense subset of M and $\mathcal{G}_{M_0} \cong M_0 \times M_0$. Moreover $\mathcal{G}_{\partial M}$ is amenable, since \mathcal{G} is (or directly). We only need to show that the vector representation is injective. This will stem from the fact that our boundary groupoid \mathcal{G} is Hausdorff and that M_0 is an open dense subset. (Recall that a space is Hausdorff if, and only if, if any two functions are the same on a dense subset, then they coincide.)

Lemma 6.2. *Let \mathcal{G} be a Hausdorff Lie groupoid with units M and M_0 be an invariant open dense subset of M such that $\mathcal{G}|_{M_0} \cong M_0 \times M_0$. Let $P \in \Psi^\infty(\mathcal{G})$. Then, if $P_x = 0$ for $x \in M_0$, then $P = 0$ on M . In particular, the vector representation π is injective.*

Proof. The first assertion is really Corollary 4.3 in [45]. We outline the argument: if $P_x = 0$, for some $x \in M_0$, $P \in \Psi^m(\mathcal{G})$ (or $P \in \Psi^0(\mathcal{G})$, or $P \in C^*(\mathcal{G})$), then by left-invariance $P_y = 0$ for all $y \in M_0$, that is, $P_{M_0} = 0$. Since P is a smooth family, \mathcal{G} is Hausdorff, and M_0 is an open dense subset of M , we have $P = 0$. In particular, the regular representation π_x is injective, for $x \in M_0$. But in this case, the vector representation π is equivalent to π_x using the isometry $\mathcal{G}_x \rightarrow M_0$. Hence, π is also injective (can also see this directly from the definition) and the second assertion follows. \square

Now let $\Omega \subset \mathbb{R}^n$ be a conical domain with no cracks. Let $\Sigma(\Omega)$ be the desingularization of Ω and $\partial'\Sigma(\Omega) \subset \partial\Sigma(\Omega)$ be the union of hyper-faces at infinity of $\Sigma(\Omega)$, corresponding to a desingularization of $\partial\Omega$, that is, $\partial'\Sigma(\Omega) = M$ as in (14). Recall that we have the identification (see Proposition 2.3)

$$\mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega) \cong H^m(M)$$

for all $m \in \mathbb{Z}_+$, where $\mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega)$ is the m -th weighted Sobolev space as in (10) and $H^m(M) = H^m(\partial'\Sigma(\Omega))$ is the Sobolev space defined using the induced Lie structure on $\partial\Omega$. The above also holds for domains with cracks, replacing Ω by the unfolded domain Ω^u .

Theorem 6.3. *Let \mathcal{G} be the groupoid (13) with units M associated to a domain with conical points $\Omega \subset \mathbb{R}^n$ without cracks. Let $\Omega^{(0)} = \{p_1, p_2, \dots, p_l\}$ be the set of conical points. Suppose $P = (P_x)_{x \in M} \in \Psi^m(\mathcal{G})$ or $P \in \Psi^0(\mathcal{G})$.*

Then, $\pi(P) : \mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega) \rightarrow \mathcal{K}_{\frac{n-1}{2}}^0(\partial\Omega)$ is Fredholm if, and only if, P is elliptic and

$$P_x : H^m(\mathbb{R}^+ \times \partial\omega_p) \rightarrow L^2(\mathbb{R}^+ \times \partial\omega_p)$$

is invertible, for any $x = (p, x'_p) \in \partial M$, with $p \in \Omega^{(0)}$, $x'_p \in \partial\omega_p$.

If Ω is the infinite straight cone with basis $\omega \subset S^{n-1}$, we have an identification $\mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega) \cong H^m(\mathbb{R}^+ \times \partial\omega, g)$, and the metric $g = (r^{-1}dr)^2 + (dx')^2$, where x' are the coordinates on S^{n-1} . So in that case, we get for $x \in \partial\omega$,

$$P_x : \mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega) \rightarrow \mathcal{K}_{\frac{n-1}{2}}^0(\partial\Omega).$$

In case Ω is a polygonal domain with cracks, we can apply Theorem 6.3 to the groupoid (17) associated to the unfolded domain Ω^u , as in Section 4.2. Let $\Omega^{(0)} = \{p_1, p_2, \dots, p_l\}$ be the set of (true) conical points of Ω which are non-crack

points, $\mathcal{C} := \{c_1, \dots, c_m\}$ be the set of singular crack points, with $c_1, \dots, c_{m'}$ the conical crack points with not empty no-cracks part, and $\mathcal{C}^u := \{c_{ji} \mid c_{ji} \text{ covers } c_j \in \mathcal{C}, i = 1, \dots, k_{c_j}\} \subset \partial(\Omega^u)$ the set of covers of cracks. The set of vertices, i.e., conical points, of Ω^u is $V^u = \Omega^{(0)} \cup \mathcal{C}^u \cup \{c_{j0}\}_{j=1, \dots, m'}$.

Corollary 6.4. *Let \mathcal{G}^u be the groupoid (17) with units M associated to a domain with conical points $\Omega \subset \mathbb{R}^n$ with cracks. Suppose $P = (P_x)_{x \in M} \in \Psi^m(\mathcal{G})$ or $P \in \overline{\Psi^0(\mathcal{G})}$. Then, $\pi(P) : \mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega^u) \rightarrow \mathcal{K}_{\frac{n-1}{2}}^0(\partial\Omega^u)$ is Fredholm if, and only if, P is elliptic and the following operators are invertible, for $x = (y, x'_y) \in \partial M$, with $y \in V^u$, $x'_y \in \partial\omega_y$:*

- $P_x : H^m(\mathbb{R}^+ \times \partial\omega_p) \rightarrow L^2(\mathbb{R}^+ \times \partial\omega_p)$ with $y = p \in \Omega^{(0)}$;
- $P_x : H^m(\mathbb{R}^+ \times \partial\omega'_{c_j}) \rightarrow L^2(\mathbb{R}^+ \times \partial\omega'_{c_j})$, with $y = c_{j0} \in \mathcal{C}^u$, $j = 1, \dots, m'$;
- $P_x : H^m(\mathbb{R}^+ \times \partial I_{c_j}^h) \rightarrow L^2(\mathbb{R}^+ \times \partial I_{c_j}^h)$ with $y = c_{jh} \in \mathcal{C}^u$, $j = 1, 1, \dots, m$, $h = 1, \dots, k_{c_j}$ and $I_{c_j}^h$ is the h -th connected component of ω_{c_j} , respectively, of ω''_{c_j} , if c_j is non-conical, respectively, c_j is a conical crack point.

Noting now that $\partial M = \bigcup_{p \in \Omega^{(0)}} \{p\} \times \partial\omega_p$ can be written as a union of closed invariant subsets associated to each conical point p_i , we can also apply Theorem 7.4 in [21]. Let $\mathcal{R}_{p_i} : \Psi^m(\mathcal{G}) \rightarrow \Psi^m(\mathcal{G}_{\{p\} \times \partial\omega_p}) = \Psi^m(\mathbb{R}^+ \times (\partial\omega_i)^2 \times \{p_i\})$. Recall that pseudodifferential operators with kernel in $\mathbb{R}^+ \times (\partial\omega_i)^2 \times \{p_i\}$ coincide with Mellin convolution operators on $\mathbb{R}^+ \times \partial\omega_{p_i}$. If $Q \in \Psi^m(\mathbb{R}^+ \times (\partial\omega_i)^2 \times \{p_i\})$, we denote by $\tilde{Q} : H^m(\mathbb{R}^+ \times \partial\omega_p) \rightarrow L^2(\mathbb{R}^+ \times \partial\omega_p)$ the induced Mellin convolution operator.

Theorem 6.5. *Let \mathcal{G} be the groupoid (13) with units M associated to a domain with conical points $\Omega \subset \mathbb{R}^n$ without cracks. Let $\Omega^{(0)} = \{p_1, p_2, \dots, p_l\}$ be the set of conical points. Suppose $P = (P_x)_{x \in M} \in \Psi^m(\mathcal{G})$ or $P \in \overline{\Psi^0(\mathcal{G})}$. Then, $\pi(P) : \mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega) \rightarrow \mathcal{K}_{\frac{n-1}{2}}^0(\partial\Omega)$ is Fredholm if, and only if, P is elliptic and*

$$\pi \circ \mathcal{R}_{p_i}(P) : H^m(\partial\omega_{p_i}) \rightarrow L^2(\partial\omega_{p_i})$$

is invertible, for any $p_i \in \Omega^{(0)}$, or, equivalently, if the Mellin convolution operator $\widetilde{\mathcal{R}_{p_i}(P)} : H^m(\mathbb{R}^+ \times \partial\omega_p) \rightarrow L^2(\mathbb{R}^+ \times \partial\omega_p)$ is an invertible operator.

There is an analogue of the result above for domains with cracks, where we replace Ω by the unfolded domain Ω^u .

We expect that Theorems 6.3 and 6.5 can be used to prove (or disprove) Fredholmness of certain integral operators arising from boundary value problems, namely in applications of the layer potentials method, generalizing the results in [46]. This goal will be pursued in a forthcoming paper.

CONCLUSION

To a conical domain Ω we associate a boundary groupoid \mathcal{G} with space of units given by a desingularization M of $\partial\Omega$. The layer potentials C^* -algebra associated to Ω is defined as the groupoid convolution algebra $C^*(\mathcal{G})$. For straight cones, this C^* -algebra is Morita equivalent to an algebra of Wiener-Hopf operators. In two dimensions, we allow domains with ramified cracks, using the notion of unfolded domain.

We study the structure of the boundary groupoid and its C^* -algebra, which is identified with an ideal in the norm closure of order 0 pseudodifferential operators on \mathcal{G} . The invariant subsets of \mathcal{G} are given by the smooth part $\Omega_0 \subset \partial\Omega$, and we get also an invariant subset for each conical point p given by $\{p\} \times \partial\omega_p$, with ω_p the basis of the local cone at p , so that

$$\mathcal{G}_{\Omega_0} = \Omega_0 \times \Omega_0, \quad \mathcal{G}_{\{p\} \times \partial\omega_p} = (\partial\omega_p \times \partial\omega_p) \times \mathbb{R}^+ \times \{p\}.$$

It is seen that, up to equivalence, the boundary groupoid only depends on the number of conical points of Ω , yielding Morita equivalent C^* -algebras. Moreover, we compute the K -groups of the layer potentials C^* -algebra, finding that

$$K_0(C^*(\mathcal{G})) = 0, \quad K_1(C^*(\mathcal{G})) = \mathbb{Z}^{l-1},$$

where l is the number of (true) conical points of Ω . As for pseudodifferential operators on \mathcal{G} , at the interior we get a pseudodifferential operator on $\Omega_0 \subset \partial\Omega$ smooth, and for each conical point p we have a Mellin convolution operator Q_p on $\mathbb{R}^+ \times \partial\omega_p$. Our final result is a Fredholm criterion, which yields that for $P \in \Psi^0(\mathcal{G})$ or $P \in \Psi^\infty(\mathcal{G})$, with $P = (P_x)$, then the vector representation $\pi(P)$ on $C_c^\infty(M)$,

$$\pi(P) : \mathcal{K}_{\frac{n-1}{2}}^m(\partial\Omega) \rightarrow \mathcal{K}_{\frac{n-1}{2}}^0(\partial\Omega)$$

is a Fredholm operator between suitable weighted Sobolev spaces if, and only if, P is elliptic and the following family of operators is invertible, for $x \in \partial M$,

$$P_x : H^m(\mathbb{R}^+ \times \partial\omega_p) \rightarrow L^2(\mathbb{R}^+ \times \partial\omega_p).$$

Alternatively, we can replace the second condition above by invertibility of the Mellin convolution operators Q_p , for each conical point p .

We expect our results to apply to the study of Fredholmness and compactness of boundary operators related to applications of the method of layer potentials for boundary value problems on conical domains.

REFERENCES

- [1] J. Aastrup, S. Melo, B. Monthubert, and E. Schrohe. Boutet de Monvel's calculus and groupoids I. *J. Noncommut. Geom.*, 4(3):313–329, 2010.
- [2] A. Alldridge and T. Johansen. An index theorem for Wiener-Hopf operators. *Adv. Math.*, 218(1):163–201, 2008.
- [3] B. Ammann, R. Lauter, and V. Nistor. On the geometry of Riemannian manifolds with a Lie structure at infinity. *Int. J. Math. Math. Sci.*, 2004(1-4):161–193, 2004.
- [4] B. Ammann, R. Lauter, and V. Nistor. Pseudodifferential operators on manifolds with a Lie structure at infinity. *Ann. of Math. (2)*, 165(3):717–747, 2007.
- [5] C. Băcuță, A. Mazzucato, V. Nistor, and L. Zikatanov. Interface and mixed boundary value problems on n -dimensional polyhedral domains. *Doc. Math.*, 15:687–745, 2010.
- [6] A. Cannas da Silva and A. Weinstein. *Geometric models for noncommutative algebras*, volume 10 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI, 1999.
- [7] A. Connes. *Noncommutative Geometry*. Academic Press, New York London, 1994.
- [8] H. O. Cordes. On the technique of comparison algebra for elliptic boundary problems on non-compact manifolds. In *Operator theory: operator algebras and applications, Part 1 (Durham, NH, 1988)*, volume 51 of *Proc. Sympos. Pure Math.*, pages 113–130. Amer. Math. Soc., Providence, RI, 1990.
- [9] C. Debord and J.-M. Lescure. K -duality for pseudomanifolds with an isolated singularity. *C. R. Math. Acad. Sci. Paris*, 336(7):577–580, 2003.
- [10] C. Debord and J.-M. Lescure. K -duality for pseudomanifolds with isolated singularities. *J. Funct. Anal.*, 219(1):109–133, 2005.

- [11] C. Debord, J.-M. Lescure, and V. Nistor. Groupoids and an index theorem for conical pseudo-manifolds. *J. Reine Angew. Math.*, 628:1–35, 2009.
- [12] Y. Egorov and B.-W. Schulze. *Pseudo-differential operators, singularities, applications*, volume 93 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.
- [13] J. Elschner. The double layer potential operator over polyhedral domains. I. Solvability in weighted Sobolev spaces. *Appl. Anal.*, 45(1-4):117–134, 1992.
- [14] E. Fabes, M. Jodeit, and J. Lewis. Double layer potentials for domains with corners and edges. *Indiana Univ. Math. J.*, 26(1):95–114, 1977.
- [15] E. Fabes, M. Jodeit, and N. Rivière. Potential techniques for boundary value problems on C^1 -domains. *Acta Math.*, 141(3-4):165–186, 1978.
- [16] G. Folland. *Introduction to partial differential equations*. Princeton University Press, Princeton, NJ, second edition, 1995.
- [17] M. Karoubi. Homologie cyclique et K -théorie. *Astérisque*, (149):147, 1987.
- [18] V. Kondrat'ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Moskov. Mat. Obšč.*, 16:209–292, 1967.
- [19] R. Kress. *Linear integral equations*, volume 82 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1999.
- [20] R. Lauter, B. Monthubert, and V. Nistor. Pseudodifferential analysis on continuous family groupoids. *Doc. Math.*, 5:625–655 (electronic), 2000.
- [21] R. Lauter and V. Nistor. Analysis of geometric operators on open manifolds: a groupoid approach. In *Quantization of singular symplectic quotients*, volume 198 of *Progr. Math.*, pages 181–229. Birkhäuser, Basel, 2001.
- [22] P.Y Le Gall and B. Monthubert. K -theory of the indicial algebra of a manifold with corners. *K-Theory*, 23(2):105–113, 2001.
- [23] J.-M. Lescure. Elliptic symbols, elliptic operators and Poincaré duality on conical pseudo-manifolds. *J. K-Theory*, 4(2):263–297, 2009.
- [24] J. Lewis. Layer potentials for elastostatics and hydrostatics in curvilinear polygonal domains. *Trans. Amer. Math. Soc.*, 320(1):53–76, 1990.
- [25] J. Lewis and C. Parenti. Pseudodifferential operators of Mellin type. *Comm. Partial Differential Equations*, 8(5):477–544, 1983.
- [26] H. Li, A. Mazzucato, and V. Nistor. Analysis of the finite element method for transmission/mixed boundary value problems on general polygonal domains. *Electron. Trans. Numer. Anal.*, 37:41–69, 2010.
- [27] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.
- [28] V. Maz'ya. Boundary integral equations. In *Analysis, IV*, volume 27 of *Encyclopaedia Math. Sci.*, pages 127–222. Springer, Berlin, 1991.
- [29] A. Mazzucato and V. Nistor. Well-posedness and regularity for the elasticity equation with mixed boundary conditions on polyhedral domains and domains with cracks. *Arch. Ration. Mech. Anal.*, 195(1):25–73, 2010.
- [30] R. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993.
- [31] R. Melrose. *Geometric scattering theory*. Stanford Lectures. Cambridge University Press, Cambridge, 1995.
- [32] R. Melrose and V. Nistor. K -theory of C^* -algebras of b -pseudodifferential operators. *Geom. Funct. Anal.*, 8:88–122, 1998.
- [33] D. Mitrea and I. Mitrea. On the Besov regularity of conformal maps and layer potentials on nonsmooth domains. *J. Funct. Anal.*, 201(2):380–429, 2003.
- [34] I. Mitrea. On the spectra of elastostatic and hydrostatic layer potentials on curvilinear polygons. *J. Fourier Anal. Appl.*, 8(5):443–487, 2002.
- [35] M. Mitrea and V. Nistor. Boundary value problems and layer potentials on manifolds with cylindrical ends. *Czechoslovak Math. J.*, 57(132)(4):1151–1197, 2007.
- [36] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.
- [37] B. Monthubert. Groupoids of manifolds with corners and index theory. In *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, volume 282 of *Contemp. Math.*, pages 147–157. Amer. Math. Soc., Providence, RI, 2001.

- [38] B. Monthubert. Groupoids and pseudodifferential calculus on manifolds with corners. *J. Funct. Anal.*, 199:243–286, 2003.
- [39] B. Monthubert and V. Nistor. A topological index theorem for manifolds with corners. Preprint, 2011.
- [40] B. Monthubert and F. Pierrot. Indice analytique et groupoïdes de Lie. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(2):193–198, 1997.
- [41] S. Moroianu. K-theory of suspended pseudo-differential operators. *K-Theory*, 28(2):167–181, 2003.
- [42] P. Muhly and J. Renault. C^* -algebras of multivariate Wiener-Hopf operators. *Trans. Amer. Math. Soc.*, 274 (1):1–44, 1982.
- [43] P. Muhly, J. Renault, and D. Williams. Equivalence and isomorphism for groupoid C^* -algebras. *J. Operator Theory*, 17:3–22, 1987.
- [44] F. Nicola. K-theory of SG-pseudodifferential algebras. *Proc. Amer. Math. Soc.*, 131(9):2841–2848, 2003.
- [45] V. Nistor. Pseudodifferential operators on non-compact manifolds and analysis on polyedral domains. In Booss B., G. Grubb, and K. P. Wojciechowski, editors, *Spectral geometry of manifolds with boundary and decomposition of manifolds*, volume 366 of *Contemporary Mathematics*, pages 307–328, Rhode Island, 2005. Amer. Math. Soc.
- [46] V. Nistor and Y. Qiao. Single and double layer potentials on domains with conical points I: Straight cones. Preprint, 2010.
- [47] V. Nistor, A. Weinstein, and P. Xu. Pseudodifferential operators on differential groupoids. *Pacific J. Math.*, 189(1):117–152, 1999.
- [48] J. Renault. *A groupoid approach to C^* -algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.
- [49] M. Rørdam, F. Larsen, and N. Laustsen. *An introduction to K-theory for C^* -algebras*, volume 49 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2000.
- [50] E. Schrohe. Spaces of weighted symbols and weighted Sobolev spaces on manifolds. In *Pseudo-Differential Operators. Proceedings Oberwolfach 1986*, volume 1256 of *Springer Lecture Notes in Mathematics*, pages 360–377, 1987.
- [51] E. Schrohe and B.-W. Schulze. Boundary value problems in Boutet de Monvel’s algebra for manifolds with conical singularities. I. In *Pseudo-differential calculus and mathematical physics*, volume 5 of *Math. Top.*, pages 97–209. Akademie Verlag, Berlin, 1994.
- [52] E. Schrohe and B.-W. Schulze. Boundary value problems in Boutet de Monvel’s algebra for manifolds with conical singularities. II. In *Boundary value problems, Schrödinger operators, deformation quantization*, volume 8 of *Math. Top.*, pages 70–205. Akademie Verlag, Berlin, 1995.
- [53] B.-W. Schulze. *Boundary value problems and singular pseudo-differential operators*. Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester, 1998.

C. CARVALHO, INSTITUTO SUPERIOR TÉCNICO, MATH. DEPT., UTL, AV. ROVISCO PAIS, 1049-001 LISBON, PORTUGAL

E-mail address: ccarv@math.ist.utl.pt

Y. QIAO, CHERN INSTITUTE OF MATHEMATICS, NANKAI UNIVERSITY, TIANJIN 300071, PEOPLE’S REPUBLIC OF CHINA

E-mail address: fishqiao@gmail.com